HW2 Solutions

1.3

(8) $w = -u + 2v$, $\quad x = -2u + 2v$

$y = -2u + 3.5v$, $\quad z = -3u + 4v$.

(9) $x_1 \begin{pmatrix} 0 \\ 4 \\ -1 \end{pmatrix} + x_2 \begin{pmatrix} 1 \\ 6 \\ 3 \end{pmatrix} + x_3 \begin{pmatrix} 5 \\ -1 \\ -8 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$

(14) $A = \begin{bmatrix} 1 & -2 & -6 \\ 0 & 3 & 7 \\ 1 & -2 & 5 \end{bmatrix}$, $\vec{b} = \begin{pmatrix} 11 \\ -5 \\ 9 \end{pmatrix}$. To find if $\vec{b}$ is a linear combination of columns of A? For this, we form the augmented matrix.

$$\left[\begin{array}{ccc:c} 1 & -2 & -6 & 11 \\ 0 & 3 & 7 & -5 \\ 1 & -2 & 5 & 9 \end{array}\right] \underset{\sim}{R_3' = R_3 - R_1} \left[\begin{array}{ccc:c} 1 & -2 & -6 & 11 \\ 0 & 3 & 7 & -5 \\ 0 & 0 & 11 & -2 \end{array}\right].$$

Note that the system is consistent. and hence $\vec{b}$ can be expressed as a linear combination of columns of A.

(22) We start with any inconsistent system, for example,

$$\left[\begin{array}{ccc:c} 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{array}\right].$$

We now try to make the zero entries to non-zero entries by row operations.

$$\left[\begin{array}{ccc:c} 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{array}\right] \underset{R_3' = R_3 + R_1}{\overset{R_2' = R_2 + R_1}{\sim}} \left[\begin{array}{ccc:c} 1 & 1 & 1 & 0 \\ 1 & 2 & 2 & 0 \\ 1 & 1 & 1 & 1 \end{array}\right]$$

$\therefore A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 2 \\ 1 & 1 & 1 \end{bmatrix}$, $\vec{b} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$

(24) (a) True ; by definition of vectors in $\mathbb{R}^n$.

(b) True ; $(\vec{u}-\vec{v})+\vec{v} = u + (-v+v)$ (by associativity of vector addition)

$= \vec{u} + \vec{0}$

$= \vec{u}$

(c) False ; by definition of linear combination.

(d) True ; The line through the origin $\vec{0}$ and $\vec{u}$ is all the vectors of the form $k\vec{u}$ where $k \in \mathbb{R}$. but $k\vec{u}$ can be seen as $k\vec{u} + 0 \cdot \vec{v}$, which is a linear combination of $\vec{u}$ and $\vec{v}$ and thus belongs to $\text{Span}\{\vec{u}, \vec{v}\}$.

(e) True ; Asking for $[\vec{a}_1 \ \vec{a}_2 \ \vec{a}_3 \mid \vec{b}]$ to have a solution, means asking if $x_1\vec{a}_1 + x_2\vec{a}_2 + x_3\vec{a}_3 = \vec{b}$ has a solution. But $x_1\vec{a}_1 + x_2\vec{a}_2 + x_3\vec{a}_3$ is a linear combination of $\vec{a}_1, \vec{a}_2, \vec{a}_3$. Hence it is equivalent ~~to ask~~ if $\vec{b}$ is in $\text{Span}\{\vec{a}_1, \vec{a}_2, \vec{a}_3\}$.

1.4

(8) $\cancel{4z_1 - 4z_2 - 5z_3 + 3z_4 = 4}$
$\cancel{-2z_1 + 5z_2 + 4z_3 = 13}$

$$\begin{bmatrix} 4 & -4 & -5 & 3 \\ -2 & 5 & 4 & 0 \end{bmatrix}\begin{pmatrix} z_1 \\ z_2 \\ z_3 \\ z_4 \end{pmatrix} = \begin{pmatrix} 4 \\ 13 \end{pmatrix}$$

(9) $x_1\begin{pmatrix} 3 \\ 0 \end{pmatrix} + x_2\begin{pmatrix} 1 \\ 1 \end{pmatrix} + x_3\begin{pmatrix} -5 \\ 4 \end{pmatrix} = \begin{pmatrix} 9 \\ 0 \end{pmatrix}$; $\begin{bmatrix} 3 & 1 & -5 \\ 0 & 1 & 4 \end{bmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 9 \\ 0 \end{pmatrix}$.

(15) $$\left[\begin{array}{cc|c} 2 & -1 & b_1 \\ -6 & 3 & b_2 \end{array}\right] \overset{R_2' = R_2 + 3R_1}{\sim} \begin{bmatrix} 2 & -1 & b_1 \\ 0 & 0 & b_2 + 3b_1 \end{bmatrix}$$

So the system has a solution as long as $b_2 + 3b_1 = 0$ i.e if $b_2 = -3b_1$ i.e if $\vec{b} = \begin{pmatrix} b_1 \\ -3b_1 \end{pmatrix}$

Let $b_1 = k$ so $\vec{b} = k\begin{pmatrix} 1 \\ -3 \end{pmatrix}$; k is in $\mathbb{R}$.

So the system $A\vec{x} = \vec{b}$ has a solution if and only if $\vec{b}$ has the form $k\begin{pmatrix} 1 \\ -3 \end{pmatrix}$ where k is in $\mathbb{R}$.

(22) $$\underbrace{\begin{bmatrix} 0 & 0 & 4 \\ 0 & -3 & -1 \\ -2 & 8 & -5 \end{bmatrix}}_{A} \underset{R_3' = R_1}{\overset{R_1' = R_3}{\sim}} \begin{bmatrix} \boxed{-2} & 8 & -5 \\ 0 & \boxed{-3} & -1 \\ 0 & 0 & \boxed{4} \end{bmatrix}.$$

We clearly see that we have 3 pivot positions, and hence the system $A\vec{x} = \vec{b}$ is consistent for every $\vec{b}$ in $\mathbb{R}^3$.

Hence columns of A span $\mathbb{R}^3$.

24 (a) True ; correspondence between matrix equation and vector equations explained in the class.

(b) True ; ~~The~~ A general linear combination of $\vec{a_1}, \vec{a_2}, \dots, \vec{a_n}$ looks like

$c_1\vec{a_1} + c_2\vec{a_2} + \dots + c_n\vec{a_n}$ for some scalars

$c_1, c_2, \dots, c_n$ in $\mathbb{R}$.

Let $A = \left[(\vec{a_1}) (\vec{a_2}) \dots (\vec{a_n}) \right]$ be the matrix

whose columns are $\vec{a_1}, \dots, \vec{a_n}$ and let

$\vec{x} = \begin{pmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{pmatrix}$; Then our linear combination

equals $A.\vec{x}$.

(c) True ; explained in class (but you should explain it during exams)

(d) True

(e) False. If it has pivot positions in every row, then it can brought in a consistent echelon form and hence ~~solvable~~ can be solved.

(f) True ; If colums of A do not span $\mathbb{R}^m$, then it does not have pivot position in some row and then that row can be reduced to all zero row; and that the entry in the $\vec{b}$ column might be non-zero; hence inconsistent.