Math 225 R2

Exam 2

Name: Solutions

Total:

1. (25 points) Let $A = \begin{pmatrix} 1 & 3 & 0 & 2 \\ -2 & -5 & 7 & 4 \\ 3 & 5 & 2 & 1 \\ 1 & -1 & 2 & -3 \end{pmatrix}$.

Find $|A|$ and conclude if columns of A form a basis of Col(A) ? If not, then find a basis of Col(A) and extend it to a basis of R^4.

$$\begin{vmatrix} 1 & 3 & 0 & 2 \\ -2 & -5 & 7 & 4 \\ 3 & 5 & 2 & 1 \\ 1 & -1 & 2 & -3 \end{vmatrix} = \begin{vmatrix} 1 & 3 & 0 & 2 \\ 0 & 1 & 7 & 8 \\ 0 & -4 & 2 & -5 \\ 0 & -4 & 2 & -5 \end{vmatrix} = \begin{vmatrix} 1 & 3 & 0 & 2 \\ 0 & 1 & 7 & 8 \\ 0 & -4 & 2 & -5 \\ 0 & 0 & 0 & 0 \end{vmatrix} = 0$$

Hence columns of A do not form a basis of Col A.

$$\left[\begin{array}{cccc|cccc} 1 & 3 & 0 & 2 & 1 & 0 & 0 & 0 \\ -2 & -5 & 7 & 4 & 0 & 1 & 0 & 0 \\ 3 & 5 & 2 & 1 & 0 & 0 & 1 & 0 \\ 1 & -1 & 2 & -3 & 0 & 0 & 0 & 1 \end{array}\right] \sim \left[\begin{array}{cccc|cccc} 1 & 3 & 0 & 2 & 1 & 0 & 0 & 0 \\ 0 & 1 & 7 & 8 & 2 & 1 & 0 & 0 \\ 0 & -4 & 2 & -5 & -3 & 0 & 1 & 0 \\ 0 & -4 & 2 & -5 & -1 & 0 & 0 & 1 \end{array}\right] \sim$$

$$\left[\begin{array}{cccc|cccc} 1 & 3 & 0 & 2 & 1 & 0 & 0 & 0 \\ 0 & 1 & 7 & 8 & 2 & 1 & 0 & 0 \\ 0 & -4 & 2 & -5 & -3 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 2 & 0 & -1 & 1 \end{array}\right] \sim \left[\begin{array}{cccc|cccc} \boxed{1} & 3 & 0 & 2 & 1 & 0 & 0 & 0 \\ 0 & \boxed{1} & 7 & 8 & 2 & 1 & 0 & 0 \\ 0 & 0 & \boxed{30} & 27 & 5 & 4 & 1 & 0 \\ 0 & 0 & 0 & 0 & \boxed{2} & 0 & -1 & 1 \end{array}\right]$$

Hence an extension of basis of Col A to $\mathbb{R}^4$ is

$$\left\{ \begin{pmatrix} 1 \\ -2 \\ 3 \\ 1 \end{pmatrix}, \begin{pmatrix} 3 \\ -5 \\ 5 \\ -1 \end{pmatrix}, \begin{pmatrix} 0 \\ 7 \\ 2 \\ 2 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} \right\}.$$

2. (15 points) Let A be an $n \times n$ invertible matrix.
(i) Show that $|adj(A)| = |A|^{n-1}$.
(ii) Show that $adj(A)$ is invertible and $(adj(A))^{-1} = \frac{1}{(det A)} A$.

(i) $A^{-1} = \frac{1}{|A|} \cdot (adj\ A)$

$\therefore (|A|) A^{-1} = adj\ A$

$\therefore |adj\ A| = |A|^{n} \cdot |A^{-1}| = |A|^{n} \cdot \frac{1}{|A|} = |A|^{n-1}$ $(\because |A| \neq 0)$

(ii) Since $|A| \neq 0$

$|adj\ A| = |A|^{n-1} \neq 0$ Hence $adj\ A$ is invertible.

We check if $(adj\ A) \cdot (\frac{1}{|A|} \cdot A) = I$

$L.H.S = |A| \cdot (A^{-1}) \cdot \frac{1}{|A|} A$

$= (|A| \cdot \frac{1}{|A|}) A^{-1} A$

$= I$.

3. (20 points) Let $A = \begin{pmatrix} 1 & 2 & 0 & 1 \\ 2 & 4 & 0 & 2 \\ 4 & 3 & 1 & 3 \end{pmatrix}$. Explicitly express Nul(A), Col(A), Row(A) and find a basis for each. Find rank(A), dim Nul(A), dim Col(A), dim Row(A).

$$\begin{pmatrix} 1 & 2 & 0 & 1 \\ 2 & 4 & 0 & 2 \\ 4 & 3 & 1 & 3 \end{pmatrix} \sim \begin{pmatrix} 1 & 2 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & -5 & 1 & -1 \end{pmatrix} \sim \begin{pmatrix} 1 & 2 & 0 & 1 \\ 0 & -5 & 1 & -1 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$\therefore$ $x_1 + 2x_2 + x_4 = 0$

$-5x_2 + x_3 - x_4 = 0$

Let $\cancel{x_3, x_4}$ $x_2 = k$, $x_4 = l$

$\therefore$ $x_3 = 5k + l$

$x_1 = -2k - l$

$\therefore$ $\vec{x} = \begin{pmatrix} -2k - l \\ k \\ 5k + l \\ l \end{pmatrix} = k\begin{pmatrix} -2 \\ 1 \\ 5 \\ 0 \end{pmatrix} + l\begin{pmatrix} -1 \\ 0 \\ 1 \\ 1 \end{pmatrix}$

$\therefore$ $\text{Nul}(A) = \text{Span}\left\{\begin{pmatrix} -2 \\ 1 \\ 5 \\ 0 \end{pmatrix}, \begin{pmatrix} -1 \\ 0 \\ 1 \\ 1 \end{pmatrix}\right\}$

$\text{Col}(A) = \text{Span}\left\{\begin{pmatrix} 1 \\ 2 \\ 4 \end{pmatrix}, \begin{pmatrix} 2 \\ 4 \\ 3 \end{pmatrix}\right\}$

$\text{Row}\,A = \text{Span}\left\{\begin{pmatrix} 1 \\ 2 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ -5 \\ 1 \\ -1 \end{pmatrix}\right\}$

the set which span here also forms basis of each of them.

rank(A) = dim Col(A) = 2

dim Row(A) = 2

dim Nul(A) = 2

4. (20 points) State which of the following is True/False. Justify your answer.

(i) Let A be an $n \times n$ matrix. If $|A| = 0$, then $\text{rank}(A) < n$.
(ii) Let A be an $m \times n$ matrix. Then $\text{rank}(A) \leq \min\{m, n\}$.
(iii) Let A be an $n \times n$ matrix. Then $|-A| = -|A|$.
(iv) If B is an echelon form of A, then $\text{Col}(A) = \text{Col}(B)$.

(i) True; since A will have $< n$ pivots

(ii) True; A can have at most ~~[illegible]~~ m and at most n pivots $\therefore$ $\text{rank}(A) \leq \min\{m, n\}$.

(iii) False; if ~~A~~ n is even, then $|-A| = (-1)^n |A| = |A|$.

(iv) False; let $A = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$, $B = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

$A \sim B$

but $\text{col}(A) = \left\{ \begin{pmatrix} k \\ 2k \end{pmatrix} : k \in \mathbb{R} \right\}$

$\text{col}(B) = \left\{ \begin{pmatrix} k \\ 0 \end{pmatrix} : k \in \mathbb{R} \right\}$

clearly $\text{col}(A) \neq \text{col}(B)$.

5. (20 points) Let $S = \{\vec{x} \in R^4 : x_1 + x_2 = x_3 + x_4\}$. Is S a subspace of R^4 ? If yes, then find a basis of S and $\dim(S)$.

Yes, S is a subspace of $\mathbb{R}^4$

since $S = \{\vec{x} \in \mathbb{R}^4 \ \ x_1 + x_2 - x_3 - x_4 = 0\}$

i.e $S = \text{Nul}\left(\begin{bmatrix} 1 & 1 & -1 & -1 \end{bmatrix}\right)$

Let $x_2 = k$, $x_3 = l$, $x_4 = m$

$\therefore x_1 = -k + l + m$

$\therefore \vec{x} = \begin{pmatrix} -k+l+m \\ k \\ l \\ m \end{pmatrix} = k\begin{pmatrix} -1 \\ 1 \\ 0 \\ 0 \end{pmatrix} + l\begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \end{pmatrix} + m\begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \end{pmatrix}$

$\therefore$ basis of S: $\left\{\begin{pmatrix} -1 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \end{pmatrix}\right\}$.

$\dim S = 3$.