Math 225 R2

Exam 1

Name: Solutions

Total:

1.(15 points) Let $A=\begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$, $B=\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$, $C=\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$

Compute $(A^{225} - B^{2009})C^T$.

$$A^2 = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \cdot \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}$$

$$A^3 = A^2 \cdot A = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix} \cdot \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\therefore A^3 = 0 \implies A^{225} = A^{222} \cdot A^3 = A^{222} \cdot 0 = 0$$

$$B^2 = B \cdot B = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} = B$$

$$\therefore B^2 = B \implies B^{2009} = B.$$

$$\therefore (A^{225} - B^{2009})C^T = (0 - B)C^T = -BC^T$$

$$= -\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

$$= -\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} -1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

2.(15 points) Let the consumption matrix for an economy with two sectors be $C=\begin{pmatrix} 0.2 & 0.4 \\ 0.6 & 0.4 \end{pmatrix}$

Determine the production vector $\vec{x}$ so as to satisfy the final demand of $\begin{pmatrix} 20 \\ 30 \end{pmatrix}$ using the Leontief Input-Output model.

$$\vec{x} = (I - C)^{-1} \cdot \vec{d} \quad \text{where} \quad C = \begin{bmatrix} 0.2 & 0.4 \\ 0.6 & 0.4 \end{bmatrix}, \quad \vec{d} = \begin{pmatrix} 20 \\ 30 \end{pmatrix}.$$

$$(I - C) = \begin{bmatrix} 0.8 & -0.4 \\ -0.6 & 0.6 \end{bmatrix}$$

$$(I - C)^{-1} = \frac{1}{(0.8)(0.6) - (-0.6)(-0.4)} \begin{bmatrix} 0.6 & 0.4 \\ 0.6 & 0.8 \end{bmatrix}$$

$$= \frac{1}{0.24} \begin{bmatrix} 0.6 & 0.4 \\ 0.6 & 0.8 \end{bmatrix}$$

$$= \begin{bmatrix} 10/4 & 5/3 \\ 10/4 & 10/3 \end{bmatrix}$$

$$\therefore \vec{x} = (I - C)^{-1} \cdot \vec{d} = \begin{bmatrix} 10/4 & 5/3 \\ 10/4 & 10/3 \end{bmatrix} \begin{pmatrix} 20 \\ 30 \end{pmatrix} = \begin{pmatrix} 100 \\ 150 \end{pmatrix}.$$

3.(25 points) Let $A = \begin{pmatrix} 1 & -2 & -1 \\ -1 & 5 & 6 \\ 5 & -4 & 5 \end{pmatrix}$

(*i*) Compute A^{-1}, if it exists.

(*ii*) Do columns of A span $\mathbb{R}^3$?. If yes, explain why. If no, then find $\vec{b} \in \mathbb{R}^3$ such that $\vec{b} \notin$ Span{columns of A}.

(*iii*) What can you say about the linear independence of the rows of A ?

(i) $\left[\begin{array}{ccc|ccc} 1 & -2 & -1 & 1 & 0 & 0 \\ -1 & 5 & 6 & 0 & 1 & 0 \\ 5 & -4 & 5 & 0 & 0 & 1 \end{array}\right] \underset{R_3' = R_3 - 5R_1}{\overset{R_2' = R_2 + R_1}{\sim}} \left[\begin{array}{ccc|ccc} 1 & -2 & -1 & 1 & 0 & 0 \\ 0 & 3 & 5 & 1 & 1 & 0 \\ 0 & 6 & 10 & -5 & 0 & 1 \end{array}\right]$

$\underset{\sim}{R_3' = R_3 - 2R_2} \left[\begin{array}{ccc|ccc} 1 & -2 & -1 & 1 & 0 & 0 \\ 0 & 3 & 5 & 1 & 1 & 0 \\ 0 & 0 & 0 & -7 & -2 & 1 \end{array}\right]$. No pivot in the 3^{rd} row. Hence A is not invertible.

(ii) A is not invertible. and thus by IMT columns of A do not span $\mathbb{R}^3$. and so there must be at least one vector $\vec{b}$ in $\mathbb{R}^3$ which is not spanned by columns of A.

Let $\vec{b} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$.

Looking at the work done in part (i), we see that

$\left[\begin{array}{ccc|c} 1 & -2 & -1 & 1 \\ -1 & 5 & 6 & 0 \\ 5 & -4 & 5 & 0 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & -2 & -1 & 1 \\ 0 & 3 & 5 & 1 \\ 0 & 0 & 0 & -7 \end{array}\right]$ which is an inconsistent system. Hence $\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \notin$ Span{columns of A}.

(iii) By IMT, A is not invertible ~~iff~~ if and only if A^T is not invertible. Moreover rows of A are columns of A^T. Hence rows of A are linearly dependent. (by IMT again).

4.(20 points) Consider the system:

$$x_1 + 3x_2 - 5x_3 = 4$$
$$x_1 + 4x_2 - 8x_3 = 7$$
$$-3x_1 - 7x_2 + 9x_3 = -6$$

(i) Solve the above system and give a geometric explanation of the solution set.
(ii) State the solution set for the corresponding homogeneous system and give a geometric description for the same.

(i) $\left[\begin{array}{ccc|c} 1 & 3 & -5 & 4 \\ 1 & 4 & -8 & 7 \\ -3 & -7 & 9 & -6 \end{array}\right] \underset{R_3' = R_3 + 3R_1}{\overset{R_2' = R_2 - R_1}{\sim}} \left[\begin{array}{ccc|c} 1 & 3 & -5 & 4 \\ 0 & 1 & -3 & 3 \\ 0 & 2 & -6 & 6 \end{array}\right] \overset{R_3' = R_3 - 2R_2}{\sim} \left[\begin{array}{ccc|c} 1 & 3 & -5 & 4 \\ 0 & 1 & -3 & 3 \\ 0 & 0 & 0 & 0 \end{array}\right]$

$\therefore\ x_1 + 3x_2 - 5x_3 = 4$
$x_2 - 3x_3 = 3.$

Let $x_3 = k$
$\therefore\ x_2 = 3 + 3k$
$x_1 = 4 - 3x_2 + 5x_3$
$= 4 - 3(3+3k) + 5k$
$= -5 - 4k.$

$\therefore\ \vec{x} = \begin{pmatrix} -5-4k \\ 3+3k \\ k \end{pmatrix} = \begin{pmatrix} -5 \\ 3 \\ 0 \end{pmatrix} + k\begin{pmatrix} -4 \\ 3 \\ 1 \end{pmatrix}$

The solution set is a straight line passing $\begin{pmatrix} -5 \\ 3 \\ 0 \end{pmatrix}$ and parallel to $\begin{pmatrix} -4 \\ 3 \\ 1 \end{pmatrix}$.

(ii) Here we need to solve $A\vec{x} = \vec{0}$ where $A = \begin{bmatrix} 1 & 3 & -5 \\ 1 & 4 & -8 \\ -3 & -7 & 9 \end{bmatrix}$

$\begin{bmatrix} 1 & 3 & -5 \\ 1 & 4 & -8 \\ -3 & -7 & 9 \end{bmatrix} \sim \cdots \sim \begin{bmatrix} 1 & 3 & -5 \\ 0 & 1 & -3 \\ 0 & 0 & 0 \end{bmatrix}$. Thus $\vec{x} = k\begin{pmatrix} -4 \\ 3 \\ 1 \end{pmatrix}$ (check yourself!).

This is a straight line passing through the origin $\vec{0}$ and ~~parallel to~~ of $\begin{pmatrix} -4 \\ 3 \\ 1 \end{pmatrix}$.

5.(25 points) State which of the following is True/False. Justify your answer.

(*i*) Let A, B be $n \times n$ invertible matrices. Then the systems $A\vec{x} = \vec{0}$ and $B\vec{x} = \vec{0}$ are equivalent.

(*ii*) Consider the system $A\vec{x} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$, where A is a 3×3 matrix. We know that if A is invertible then the given system has a unique solution. Is the converse true ?

(*iii*) If $A\vec{x} = \vec{b}$ has a solutiuon, then Span{columns of A} = Span{columns of A, $\vec{b}$}

(*iv*) The inverse of a matrix, if exists, is unique.

(*v*) Let A and B be $n \times n$ matrices, then $(A + B)(A - B) = A^2 - B^2$.

(i) True; Since A^{-1} and B^{-1} exists, both the given systems have $\vec{x} = \vec{0}$ as their solution set.

Hence the systems are equivalent.

(ii) True; If $A\vec{x} = \begin{pmatrix}1\\1\\1\end{pmatrix}$ has a unique solution, A must have 3 pivot points and hence invertible.

(iii) True; since $\vec{b}$ is in Span{columns of A}, we have

Span{cols. of A} = Span{cols of A, $\vec{b}$}.

(iv) True; Let A have two inverses B and C

so $B = BI = \underbrace{B(AC) = (BA)C}_{\text{by associativity.}} = IC = C$

Hence $B = C$, i.e inverse is unique.

(v) False; $(A+B)(A-B) = A^2 + BA - AB - B^2 \neq A^2 - B^2$

Since BA might not equal AB.