

COLORING UNIFORM HYPERGRAPHS WITH SMALL EDGE DEGREES

ALEXANDR V. KOSTOCHKA*, MOHIT KUMBHAT and
VOJTĚCH RÖDL†

Dedicated to the 60th birthday of László Lovász

Let k be a positive integer and $n = \lfloor \log_2 k \rfloor$. We prove that there is an $\varepsilon = \varepsilon(k) > 0$ such that for sufficiently large r , every r -uniform hypergraph with maximum edge degree at most

$$\varepsilon(k) k^r \left(\frac{r}{\ln r} \right)^{\frac{n}{n+1}}$$

is k -colorable.

1. INTRODUCTION

The degree of an edge e in a hypergraph G is the number of other edges of G intersecting e , and *the maximum edge degree* of G is the maximum over the degrees of its edges. A natural question is: Which bound on the maximum edge degree of an r -uniform hypergraph G provides that G is k -colorable? The classical result in this direction belongs to Erdős and Lovász. In their seminal paper [2] (where the Lovász Local Lemma appeared), they proved the following bound.

*Research of this author is supported in part by NSF grant DMS-0650784 and by grant 06-01-00694 of the Russian Foundation for Basic Research.

†Research of this author was supported by NSF grants DMS-0300529 and DMS-0800070.

Theorem 1 [2]. *If $k, r \geq 2$, then every r -uniform hypergraph with maximum edge degree at most $\frac{1}{4}k^r$ is k -colorable.*

The proof works also for list coloring. A remarkable feature of this result is that it works for all $k, r \geq 2$, and in many cases the bound is rather close to the best possible. In particular, Erdős and Lovász [2] showed that the bound cannot be significantly improved even if we consider only hypergraphs with high girth. If we denote by $D(k, r)$ (respectively, $D(k, r, g)$) the minimum D such that there exists an r -uniform non- k -colorable hypergraph G with (girth at least g and) maximum edge degree D , then the results mentioned above can be summarized as follows. For every $g \geq 2$,

$$(1) \quad \frac{1}{4}k^r < D(k, r) \leq 20r^3k^{r-1}.$$

The upper bound in (1) was recently improved for $k < r$ by Kostochka and Rödl [5] to $r \lceil r k^{r-1} \ln k \rceil$.

Let $m(r, k)$ denote the minimum number of edges in an r -uniform hypergraph that is not k -colorable. Elaborating the proof of a lower bound on $m(r, 2)$, and using the Local Lemma, Radhakrishnan and Srinivasan [6] improved the lower bound on $D(2, r)$ for large r to

$$(2) \quad D(2, r) \geq 0.17 \cdot 2^r \sqrt{r/\ln r}.$$

The main result of this paper is the following extension of (2) to fixed k and large r .

Theorem 2. *For every integer $k \geq 2$, let $\varepsilon = \varepsilon(k) = \exp\{-4k^2\}$ and $n = n(k) = \lfloor \log_2 k \rfloor$. Then for every sufficiently large r , every r -uniform hypergraph with maximum edge degree at most $D = \varepsilon k^r \left(\frac{r}{\ln r}\right)^{\frac{n}{n+1}}$ is k -colorable. In other words, $D(k, r) > \varepsilon k^r \left(\frac{r}{\ln r}\right)^{\frac{n}{n+1}}$.*

Recall that for the class of simple hypergraphs, Kostochka and Kumbhat [4] proved recently a slightly stronger bound than Theorem 2 gives. Recall that a hypergraph G is b -simple if $|e \cap e'| \leq b$ for every distinct $e, e' \in E(G)$.

Theorem 3 [4]. *If $b \geq 1$, $k \geq 2$, and $\varepsilon > 0$ are fixed and r is sufficiently large, then every r -uniform b -simple hypergraph G with maximum edge degree at most $k^r r^{1-\varepsilon}$ is k -colorable.*

We think that the statement of Theorem 3 holds for all r -uniform hypergraphs.

The proof of Theorem 2 uses the Local Lemma, ideas of Radhakrishnan and Srinivasan [6], and the proof of the lower bound on $m(r, k)$ by Kostochka [3].

The structure of the paper is as follows. In the next section, a semi-random procedure Evolution is described and some of its simple properties are derived. In Section 3 we study the structure of so called *cause trees* arising in the analysis of Evolution. In the next two sections we define some auxiliary “bad” events and estimate their probabilities. Using the independence structure of these auxiliary events and the Local Lemma, in the final section we show that for hypergraphs satisfying the conditions of Theorem 2, with positive probability Evolution gives a proper k -coloring. This means that such a coloring exists.

2. COLORING PROCEDURE EVOLUTION AND ITS PROPERTIES

Let k, n and ε be as in the statement of the theorem. Let

$$(3) \quad r \geq \exp \{2\varepsilon^{-2(n+1)}\}.$$

Throughout the paper we will use the notation

$$(4) \quad z = \lfloor 4k^2r / \ln r \rfloor.$$

Fix some $0 < p < 2^{-k^r}$. Then there is the unique positive integer s such that $sp \leq \frac{\ln r}{(n+1)r} < (s+1)p$. Let $G = (V, E)$ be an r -uniform hypergraph with maximum edge degree at most $D = \varepsilon k^r \left(\frac{r}{\ln r}\right)^{\frac{n}{n+1}}$.

The coloring procedure Evolution described below consists of $n+1$ stages, and every stage apart from Stage 0 consists of s steps. For $1 \leq l \leq n$ and $1 \leq i \leq s$, Step $(l-1)s+i$ is the i th step in Stage l .

We also fix a linear order L on $V(G)$. Now, the procedure works as follows.

Stage 0. (O1) Color every vertex $v \in V(G)$ randomly and independently, with a color $\phi(v) \in \{0, 1, 2, \dots, k-1\}$ chosen uniformly in this set.

(O2) For every $v \in V(G)$, define the random variable $I(v)$ with the values in $\{1, 2, \dots, sn\} \cup \{\infty\}$ as follows:

$$(5) \quad \Pr \{I(v) = x\} = \begin{cases} p, & \text{if } x \in \{1, 2, \dots, sn\}; \\ 1 - psn, & \text{if } x = \infty. \end{cases}$$

Each random variable $I(v)$ is defined to be mutually independent of all other $I(w)$.

Stage l , $l = 1, \dots, n$.

STEP $i + s(l - 1)$, $1 \leq i \leq s$. Following order L , for one by one vertex $v \in V(G)$, check whether

(C1) $I(v) = (l - 1)s + i$ and

(C2) v belongs to an edge that was monochromatic, say, of color α , before Stage l , and still is monochromatic at the current moment.

If both conditions (C1) and (C2) hold, then recolor v with color $\alpha + 2^{l-1}$ (modulo k). Otherwise, do nothing with v .

Remark 1. By Condition (C1), each vertex can be recolored at most once.

Remark 2. As it follows from the description of the procedure, every step consists of $|V(G)|$ smaller steps (one per vertex).

Lemma 1. For every $w, q \geq 1$, every set $W \subseteq V$ with $|W| = w$, and every set $Q \subseteq \{1, 2, \dots, sn\}$ with $|Q| = q$, the probability that for each vertex $v \in W$, $I(v) \in Q$ is at most $(qp)^w$.

Proof. For every vertex $v \in V(G)$ and every $1 \leq l \leq n$ and $1 \leq i \leq s$, $\Pr \{I(v) = s(l - 1) + i\} = p$. Therefore, the probability that $I(v) \in Q$ is at most qp . The mutual independence of all $I(v)$ yields the lemma. ■

For an edge $e \in E$ and $1 \leq l \leq n$, let

$$(6) \quad M(e, l) = \{v \in e : I(v) \leq sl\}.$$

Lemma 2. For every $e \in E$ and $1 \leq l \leq n$,

$$\Pr \{|M(e, l)| \geq z\} \leq \varepsilon^{0.5r}.$$

Proof. It is enough to prove the lemma for $l = n$. Observe that $|M(e, n)| \geq z$ means that there exists a set $Z \subset e$ with $|Z| = z$ such that $I(v) \leq sn$ for every $v \in Z$. Hence by Lemma 1 applied with $Q = [sn]$ and $W = Z$, this probability is at most

$$\binom{r}{z} (nsp)^z \leq \left(\frac{er}{z}\right)^z \left(\frac{n \ln r}{(n+1)r}\right)^z \leq \left(\frac{ne \ln r}{z(n+1)}\right)^z.$$

Since r is large and $z = \lfloor 4k^2r / \ln r \rfloor > n$,

$$\frac{ne \ln r}{z(n+1)} \leq \frac{e \ln r}{z+1} \leq \frac{e \ln^2 r}{4k^2r} \leq r^{-0.6}.$$

Thus

$$\left(\frac{ne \ln r}{z(n+1)}\right)^z \leq (r^{-0.6})^{(4k^2r/\ln r)-1} < e^{-2k^2r} = \varepsilon^{0.5r}. \quad \blacksquare$$

Lemma 3. *If a vertex is of color α at the end of Stage l , $l \geq 1$, then at the end of Stage 0 it can be colored only with colors $\alpha, \alpha - 2^0, \alpha - 2^1, \dots, \alpha - 2^{l-1}$ (modulo k).*

Proof. By Remark 1, every vertex can be recolored at most once and by definition, a vertex of color β can be recolored during Stage j only with color $\beta + 2^{j-1}$ (modulo k). \blacksquare

Definition [Blaming edges]. If an edge e_0 becomes monochromatic of color α during Stage l , then it must contain at the end of Stage 0 a vertex of color $\alpha - 2^{l-1}$. Suppose that at the end of Stage 0 it contained vertices of colors $\alpha - 2^{l_1-1}, \dots, \alpha - 2^{l_h-1}$, where $l_h = l$ and $l_1 < l_2 < \dots < l_h$. Then for every $1 \leq j \leq h$, there exists an edge e_j and a vertex $v_j \in e_0 \cap e_j$ such that

- (a) e_j was monochromatic of color $\alpha - 2^{l_j-1}$ at the end of Stage $l_j - 1$;
- (b) v_j was recolored with α during Stage l_j and it was the last vertex of this color in e_0 recolored with α .

In this case we say that e_0 and v_j l_j -blame e_j .

Remark 3. If an edge e_0 becomes monochromatic of color α during Stage l , then all vertices of color $\alpha - 2^{l-1}$ in e_0 change their colors to α during this stage. But since in every step, the vertices of G are considered consecutively, there is the last vertex v^* of color $\alpha - 2^{l-1}$ in e_0 that changes its color. At that moment, any monochromatic edge e of color $\alpha - 2^{l-1}$ containing v^* shares only v^* with e_0 .

Remark 4. It may happen that an edge e_0 can blame more than one edge containing the same vertex v_j . On the other hand, by definition, e_0 cannot blame an edge containing another vertex $v \in e_0$ with $\phi(v) = \phi(v_j)$.

Definition [Cause trees]. If an edge e_0 is monochromatic of color α at the end of Stage l , then a *cause tree* $T = T(e_0, \alpha, l)$ is a subset of edges of G defined by induction on l as follows:

- (a) The set T always contains e_0 .
- (b) If e_0 was monochromatic of color α already after Stage 0, then $T = \{e_0\}$ for every l .
- (c) Suppose that at the end of Stage 0 edge e_0 contained vertices of colors $\alpha - 2^{l_1-1}, \dots, \alpha - 2^{l_h-1}$, where $l_h \leq l$ and $l_1 < l_2 < \dots < l_h$. Suppose further that for $j = 1, \dots, h$, edge e_0 l_j -blames edge e_j . Then

$$T = T(e_0, \alpha, l) = \{e_0\} \cup \bigcup_{j=1}^h T(e_j, \alpha - 2^{l_j-1}, l_j - 1).$$

Remark 5. By Remark 4 and the definition of cause trees, it could be that in the same outcome of Evolution for the same triple (e_0, α, l) , we can construct several distinct cause trees $T = T(e_0, \alpha, l)$.

Definition [Levels of edges]. If $T = T(e_0, \alpha, l)$ is defined as above, then we also say that e_1, e_2, \dots, e_h are the *edges of level 1* of T , the edges blamed by the edges of level 1 are the *edges of level 2* of T , and so on. Thus, if an edge e of a cause tree has vertices of exactly t distinct colors at the end of Stage 0, then e blames either $t - 1$ or t other edges.

3. STRUCTURE OF CAUSE TREES

Since each vertex can be recolored at most once, each edge at different stages of Evolution can become monochromatic with at most two colors. Furthermore, if an edge e was monochromatic of a color α_1 after Stage l_1 and becomes monochromatic of a color $\alpha_2 \neq \alpha_1$ after Stage l_2 , then e has to be monochromatic of color α_1 already after Stage 0 and all vertices of e change their color to $\alpha_2 = \alpha_1 + 2^{l_2-1}$ at Stage l_2 . In this case, each cause tree for e considered after Stage l_2 has exactly one edge of level 1.

In view of this, if an edge e becomes monochromatic exactly once during Evolution, then the corresponding color α is called *the main color of e* and denoted by $\mu(e)$, and if e becomes monochromatic twice, then *the main color of e* , $\mu(e)$, is the first of these two colors.

If e is monochromatic of some color α after some Stage l , then we say that e is an *l -unlucky edge*.

Lemma 4. *If e_0 is an l -unlucky edge with a cause tree T , then the main colors of all the edges of T are distinct.*

Proof. If e and e' are edges of T , then there exist two sequences $e_0, e_1, \dots, e_q = e$ and $e'_0 = e_0, e'_1, \dots, e'_{q'} = e'$ such that e_j l_j -blames e_{j+1} for $j = 0, 1, \dots, q-1$ and e'_j l'_j -blames e'_{j+1} for $j = 0, 1, \dots, q'-1$. Furthermore, $l_0 > l_1 > \dots > l_{q-1}$, $l'_0 > l'_1 > \dots > l'_{q'-1}$, and the sequences l_0, l_1, \dots, l_q and l'_0, l'_1, \dots, l'_q are not identical. Thus, the numbers $2^{l_0-1} + 2^{l_1-1} + \dots + 2^{l_{q-1}-1}$ and $2^{l'_0-1} + 2^{l'_1-1} + \dots + 2^{l'_{q'-1}-1}$ are distinct and differ by less than k . On the other hand, by definition, the main color of e is $\alpha - 2^{l_0-1} - 2^{l_1-1} - \dots - 2^{l_{q-1}-1}$ and the main color of e' is $\alpha - 2^{l'_0-1} - 2^{l'_1-1} - \dots - 2^{l'_{q'-1}-1}$. This proves the lemma. ■

Lemma 5. *Suppose that e_0 is an l -unlucky edge with a cause tree T . If e and e' are edges of T and neither of them blames the other, then e and e' are disjoint.*

Proof. Assume that e and e' have a common vertex v and both belong to T . Then there exist two sequences $e_0, e_1, \dots, e_q = e$ and $e'_0 = e_0, e'_1, \dots, e'_{q'} = e'$ such that e_j l_j -blames e_{j+1} for $j = 0, 1, \dots, q-1$ and e'_j l'_j -blames e'_{j+1} for $j = 0, 1, \dots, q'-1$. Furthermore, $l_0 > l_1 > \dots > l_{q-1}$, $l'_0 > l'_1 > \dots > l'_{q'-1}$.

Claim 1. $l_{q-1} \neq l'_{q'-1}$.

Proof of Claim. If $l_{q-1} = l'_{q'-1}$, then e and e' both were monochromatic at the end of Stage $l_{q-1} - 1$. But by Lemma 4, their main colors differ. This proves the claim.

Thus below we can assume that $l_{q-1} < l'_{q'-1}$. It follows that e ceased to be monochromatic before e' did. In particular, the vertex $v \in e \cap e'$ was recolored from $\mu(e)$ to $\mu(e')$. This yields that

$$(7) \quad \mu(e') - \mu(e) \text{ (modulo } k \text{) is a power of 2.}$$

Claim 2. $\mu(e') - \mu(e) = 2^{l_{q-1}-1}$ modulo k .

Proof of Claim. Recall that

$$\begin{aligned} \mu(e') - \mu(e) &= (\alpha - 2'^{l'_0-1} - 2'^{l'_1-1} - \dots - 2'^{l'_{q-1}-1}) \\ &\quad - (\alpha - 2^{l_0-1} - 2^{l_1-1} - \dots - 2^{l_{q-1}-1}). \end{aligned}$$

In this expression, α cancels out and every other summand apart from $2^{l_{q-1}-1}$ is divisible by $2^{l_{q-1}}$. Together with (7), this yields the claim.

Claim 2 implies that v was recolored during Stage l_{q-1} and thus $\mu(e') = \mu(e_{q-1})$. This contradicts Lemma 4. ■

Lemma 6. *Let e_0 be an edge of G that is unlucky after Stage l . Let T be a cause tree for this event such that for every pair of edges (e_1, e_2) of T such that e_1 blames e_2 , we have $\mu(e_1) - \mu(e_2) \in \{1, 2, \dots, 2^{l-1}\}$ (modulo k). Then $|E(T)| \leq 2^l$. In particular, each cause tree has at most $2^n \leq k$ edges.*

Proof. If e_1 l_1 -blames e_2 and e_2 l_2 -blames e_3 , then $l_2 < l_1$. Thus, under conditions of the lemma, for the root e_0 and an arbitrary edge e of the tree, we have

$$\mu(e_0) - \mu(e) \in \{1, 2, \dots, 2^{l-1} + 2^{l-2} + \dots + 1\} = \{1, 2, \dots, 2^l - 1\}.$$

Now, Lemma 4 implies that T has at most $1 + (2^l - 1)$ edges. ■

Below we will analyze which subsets of edges of G can form cause trees $T(e, \alpha, l)$ for some values of e , α and l . Lemma 5 implies that every cause tree $T = T(e, \alpha, l)$ is an r -uniform hypergraph tree in the ordinary sense rooted at e . Moreover, every vertex of such a tree belongs to at most two edges of this tree. By Lemma 6, such a tree has at most k edges. In connection with this, let us fix some notation.

Definition [r -Trees and sub- r -trees]. When we say “ r -tree”, we mean an r -uniform hypergraph tree with at most k edges in which every vertex belongs to at most two edges of this tree. By a *sub- r -tree of G* we mean an r -tree that is a subhypergraph of G .

Often, we will consider *rooted r -trees*. The *root of an r -tree* will be an edge of this r -tree, and not a vertex. Given an r -tree T with a root e_0 , the *children* of e_0 are the edges adjacent to e_0 , and for $e \in E(T)$ at distance d from e_0 (in T), the *children* of e are the edges adjacent to e that are at distance $d + 1$ from e_0 . Naturally, the *descendants* of an $e \in E(T)$ are its children, children of children and so on. If e_1 is a descendant of e_2 , then e_2 is an *ancestor* of e_1 . For an r -tree T with a root e_0 and another edge e_1 of T , by $T(e_1)$ we denote the subtree of T formed by e_1 and all its descendants. We will use the following fact on sub- r -trees of r -uniform hypergraphs.

Lemma 7. *Let H be an r -uniform hypergraph with maximum edge degree at most D . Let $e_0 \in E(H)$. Then e_0 belongs to at most $(4D)^{y-1}$ sub- r -trees of H with y edges.*

Proof. Let T be a sub- r -tree of H containing e_0 with $|E(T)| = y$. Consider T as a rooted r -tree with root e_0 . Order the edges of T e_0, e_1, \dots, e_{y-1} starting from e_0 using Breadth-First search. We say that T has *type* (h_0, \dots, h_{y-2}) if for $i = 0, \dots, y-2$, edge e_i has exactly h_i children. Since $h_0 + \dots + h_{y-2} = y-1$, the number of distinct types does not exceed the number of representation of $y-1$ as the sum of $y-1$ of ordered nonnegative summands, which equals $\binom{(y-1)+(y-1)-1}{y-2} < 4^{y-1}$. When we know the type of T , then for every edge $e_i, i \geq 1$, we know the immediate ancestor (father edge). So, we can embed a tree T of a given type, edge by edge into G . Furthermore, at each step $i, i \geq 1$, we have at most D choices for our edge among the edges of G adjacent to its father edge. Thus, e_0 belongs to at most D^{y-1} r -trees of given type with y edges. Since the number of distinct types is at most 4^{y-1} , this proves the lemma. ■

4. AUXILIARY EVENTS

The goal of this section is to introduce the auxiliary events $\widetilde{W}(e_0, \alpha, T, l)$ that imply the “bad” events in Evolution and can be controlled. In the next section we estimate probabilities of these auxiliary events.

Definition [Events $W(e, \alpha, T, l)$]. Let $e \in E(G)$, $\alpha \in [k]$, $l \in \{1, \dots, n\}$ and T be a sub- r -tree of G rooted at e . Then let $W(e, \alpha, T, l)$ be the event that edge e is monochromatic of color α after Stage l of Evolution, and a cause tree for this is T .

Lemma 8. *Let $e \in E(G)$, $\alpha \in [k]$, $l \in \{1, \dots, n\}$ and T be a sub- r -tree of G rooted at e . Also, let e_1, \dots, e_q be the edges of T of the first level, i.e., the edges of T sharing a vertex with e . For $j \in [q]$, let $e \cap e_j = \{v_j\}$. Let $Q \doteq \{v_1, \dots, v_q\}$. If $W(e, \alpha, T, l)$ occurs, then the following properties hold:*

(W1) *For every $v \in e$, $\phi(v) \in \{\alpha\} \cup \phi(Q) \subseteq \{\alpha, \alpha - 2^0, \alpha - 2^1, \dots, \alpha - 2^{l-1}\}$ (modulo k).*

(W2) *For $j \in [q]$, $\phi(v_j) \neq \alpha$, and for distinct j and j' , $\phi(v_{j'}) \neq \phi(v_j)$. In particular, if $A_j = A_j(e, \phi) = \{v \in e : \phi(v) = \phi(v_j)\}$, then all sets A_j are disjoint subsets of e .*

(W3) $I(v) \leq ls$ for each $v \in \bigcup_{j=1}^q A_j$.

Moreover, for each $j \in [q]$, if v_j becomes of color α at Stage l_j , then

(W4) $\alpha - \phi(v_j) = 2^{l_j-1}$;

(W5) the event $W(e_j, \phi(v_j), T(e_j), l_j - 1)$ occurs;

(W6) for every $u \in e_j$ with $I(u) > (l_j - 1)s$, we have also $I(u) \geq I(v_j)$; and

(W7) for each $u \in A_j - v_j$, $(l_j - 1)s + 1 \leq I(u) \leq I(v_j)$.

(W8) If e was already monochromatic after Stage $l - 1$, then for each $v \in e$, $I(v) \notin [s(l - 1) + 1, sl]$.

Proof. We will prove (W1)–(W8) one by one.

(W1) holds by the definition of cause trees and Lemma 3.

By the definition of cause trees, for each $j \in [q]$, v_j is the last vertex of color $\phi(v_j)$ that changed its color to α . This implies both statements of (W2).

Since each $v \in \bigcup_{j=1}^q A_j$ has changed its color by Stage l , by condition (C1) in the definition of Evolution, (W3) follows.

(W4) also follows from the definition of Evolution.

If e_j were not monochromatic of color $\phi(v_j)$ after Stage $l_j - 1$, then v_j would not obtain color α blaming e_j . This yields (W5).

If some $u \in e_j$ would have $(l_j - 1)s < I(u) < I(v_j)$, then by the definition of Evolution, it would mean that u did not change its color before Stage l_j . Recall that by (W5) edge e_j is monochromatic after Stage $l_j - 1$. Thus if Condition (C2) holds at the moment $I(u)$, then vertex u should change its color at this moment, i.e. earlier than v_j did. If not, this means that some other vertex of e_j has already changed its color. In both cases, v_j would not blame e_j . This contradiction proves (W6).

Now (W7) follows from the facts that all vertices in A_j must change their colors in Stage l_j (in order to change it from $\phi(v_j)$ to α) and that v_j is the last vertex in A_j that changes its color.

If e were monochromatic after Stage $l - 1$, and $I(v) \in [s(l - 1) + 1, sl]$ for some $v \in e$, then v would change its color, and so $W(e, \alpha, T, l)$ would not happen. This proves (W8). ■

Unfortunately, events $W(e, \alpha, T, l)$ and $W(e', \alpha', T', l')$ can be dependent even if $V(T')$ is disjoint from $V(T)$. For example, some edge $e_0 \in E(G)$ may intersect both $V(T)$ and $V(T')$. In this case, event $W(e, \alpha, T, l)$ may prompt a vertex $v \in e_0 \cap V(T)$ to change its color. This in turn, can make

e_0 monochromatic and so prompt a vertex $v' \in e_0 \cap V(T')$ to change its color, which may affect $W(e', \alpha', T', l')$.

Therefore, for each $e_0 \in E(G)$, each sub- r -tree T of G with root e_0 and $|E(T)| \leq k$, and each color α , we will introduce the auxiliary event $\widetilde{W}(e_0, \alpha, T, l)$ that contains the event $W(e_0, \alpha, T, l)$, and in addition essentially possesses properties (W1)–(W8) above, but does not depend on the values of $\phi(u)$ and $I(u)$ for all $u \notin V(T)$. This will imply that each $W(e, \alpha, T, l)$ is independent of all $\widetilde{W}(e'_0, \alpha', T', l')$ with $V(T') \cap V(T) = \emptyset$.

We define these events by induction on the number of edges in T .

If $E(T) = \{e_0\}$, then the event $\widetilde{W}(e_0, \alpha, T, l)$ means that all of the following holds

- (i) $\phi(e_0)$ is monochromatic of color α ,
- (ii) $|M(e_0, n)| < z$, and
- (iii) $I(v) > ls$ for every $v \in e_0$.

Suppose that the event $\widetilde{W}(e_0, \alpha, T, l)$ is defined for all parameters e_0, α, T, l such that $|E(T)| < y$. Let $e_0 \in E(G)$, $\alpha \in [k]$, and T be any sub- r -tree T of G with root e_0 and y edges. Let e_1, \dots, e_q be the edges of T sharing a vertex with e_0 . For $j \in [q]$, let $e \cap e_j = \{v_j\}$. Let $Q \doteq \{v_1, \dots, v_q\}$. We say that $\widetilde{W}(e_0, \alpha, T, l)$ occurs, if either $|M(e, n)| \geq z$ for at least one $e \in E(T)$ or all of the following holds:

($\widetilde{W}1$) For every $v \in e$, $\phi(v) \in \{\alpha\} \cup \phi(Q) \subseteq \{\alpha, \alpha - 2^0, \alpha - 2^1, \dots, \alpha - 2^{l-1}\}$ (modulo k).

($\widetilde{W}2$) For $j \in [q]$, $\phi(v_j) \neq \alpha$, and for distinct j and j' , $\phi(v_{j'}) \neq \phi(v_j)$. In particular, if $A_j = A_j(e, \phi) = \{v \in e : \phi(v) = \phi(v_j)\}$, then all sets A_j are disjoint.

($\widetilde{W}3$) $I(v) \leq ls$ for each $v \in \bigcup_{j=1}^q A_j$.

Moreover, for each $j \in [q]$, if $(l_j - 1)s + 1 \leq I(v_j) \leq sl_j$, then

($\widetilde{W}4$) $\alpha - \phi(v_j) = 2^{l_j - 1}$;

($\widetilde{W}5$) event $\widetilde{W}(e_j, \phi(v_j), T(e_j), l_j - 1)$ occurs;

($\widetilde{W}6$) for every $u \in e_j$ with $I(u) > (l_j - 1)s$, we have also $I(u) \geq I(v_j)$, and

($\widetilde{W}7$) for each $u \in A_j - v_j$, $(l_j - 1)s + 1 \leq I(u) \leq I(v_j)$.

($\widetilde{W}8$) If event $\widetilde{W}(e_0, \alpha, T, l - 1)$ occurs, then for each $v \in e_0$, $I(v) \notin [s(l - 1) + 1, sl]$.

The following two lemmas justify the introduction of the events $\widetilde{W}(e, \alpha, T, l)$.

Lemma 9. *Let $e_0 \in E(G)$, $\alpha \in [k]$, $l \in \{0, \dots, n\}$ and T be a sub- r -tree of G with root e_0 . If the event $W(e_0, \alpha, T, l)$ occurs, then the event $\widetilde{W}(e_0, \alpha, T, l)$ also occurs.*

Proof. Suppose that for some values of the parameters e_0 , T , l , and α , $W(e_0, \alpha, T, l)$ occurs but $\widetilde{W}(e_0, \alpha, T, l)$ does not occur. We may assume that l is minimal with this property, i.e., that for all quadruples (e'_0, α', T', l') with $l' < l$ the lemma holds.

Let us check which of the properties in the definition of $\widetilde{W}(e_0, \alpha, T, l)$ may fail. Since $(\widetilde{W}1)$ and $(\widetilde{W}2)$ coincide with $(W1)$ and $(W2)$, respectively, they hold. For the same reason, properties $(\widetilde{W}4)$, $(\widetilde{W}6)$, and $(\widetilde{W}7)$ hold. Property $(\widetilde{W}5)$ follows from $(W5)$ and the minimality of our counterexample. Property $(\widetilde{W}3)$ follows from the fact that otherwise, by the definition of Evolution (Condition $(C1)$), some vertex in $\bigcup_{j=1}^q A_j$ would not change its color to α .

Assume finally that $(\widetilde{W}8)$ does not hold, in other words, that $\widetilde{W}(e_0, \alpha, T, l-1)$ occurs, and for some $v \in e_0$, $I(v) \in [s(l-1)+1, sl]$. By $(W8)$, this implies that $W(e_0, \alpha, T, l-1)$ does not occur, i.e., after Stage $l-1$, e_0 is not monochromatic of color α . It follows that in order e_0 to become monochromatic of color α after Stage l , we need $I(u) \in [s(l-1)+1, sl]$ for some $u \in \bigcup_{j=1}^q A_j$. On the other hand, by $(\widetilde{W}3)$ for the event $\widetilde{W}(e_0, \alpha, T, l-1)$, $I(u) \leq (l-1)s$ for each $u \in \bigcup_{j=1}^q A_j$. This contradiction finishes the proof of the lemma. ■

Lemma 10. *Let $e_0 \in E(G)$, $\alpha_0 \in [k]$, $l_0 \in \{0, \dots, n\}$, and T_0 be a sub- r -tree of G with root e_0 . Then $\widetilde{W}(e_0, \alpha_0, T_0, l_0)$ is independent of all events $\widetilde{W}(e, \alpha, T, l)$ such that $V(T) \cap V(T_0) = \emptyset$.*

Proof. By definition, the events $\widetilde{W}(e_0, \alpha_0, T_0, l_0, \psi(e_0))$ are completely defined when we know the values of $\phi(v)$ and $I(v)$ for all $v \in V(T_0)$. This yields the lemma. ■

5. PROBABILITIES OF THE AUXILIARY EVENTS

The whole proof is based on the following lemma.

Lemma 11. *Let $D := \varepsilon k^r \left(\frac{r}{\ln r}\right)^{\frac{n}{n+1}}$ and G be an r -uniform hypergraph with maximum edge degree at most D . Let $e \in E(G)$, $\alpha \in [k]$, and $0 \leq l \leq n$. Let T be a rooted sub- r -tree of G with root e . If T has y edges, then*

$$\Pr(\widetilde{W}(e, \alpha, T, l)) \leq \varepsilon D^{-y} \left(\frac{r}{\ln r}\right)^{\frac{n-l}{n+1}}.$$

Proof. We start from proving the (simple) auxiliary inequality

$$(8) \quad D^k \varepsilon^{0.5r} \leq \frac{\varepsilon}{100k}.$$

By definition and (3),

$$D = \varepsilon k^r \left(\frac{r}{\ln r}\right)^{\frac{n}{n+1}} \leq \frac{\varepsilon}{\ln r} k^r r \leq \frac{\varepsilon}{2\varepsilon^{-2(n+1)}} k^{2r} \leq \frac{\varepsilon}{2} \varepsilon^4 k^{2r}.$$

Since $\varepsilon^4 = e^{-16k^2} < \frac{1}{100k}$, (8) would follow from the fact that $(k^{2r})^k \varepsilon^{0.5r} \leq 1$, which is equivalent to

$$(k^{2r})^k \leq (e^{4k^2})^{0.5r}.$$

The last inequality is obvious.

To prove the lemma, we use induction on l . Consider first $l = 0$. If $\widetilde{W}(e, \alpha, T, 0)$ occurs, then by $(\widetilde{W}1)$, $\phi(v) = \alpha$ for each $v \in e$. Thus, in this case

$$\Pr(\widetilde{W}(e, \alpha, T, 0)) = k^{-r} = \frac{\varepsilon}{D} \left(\frac{r}{\ln r}\right)^{\frac{n}{n+1}}.$$

This proves the case $l = 0$.

Now, suppose that the lemma holds for every $l' < l$. Consider the event $\widetilde{W}(e, \alpha, T, l)$ for some $e \in E(G)$, an r -tree T with y edges rooted at e , and $\alpha \in [k]$.

Let $X(T)$ denote the event that for at least one $e' \in E(T)$, the set $M(e', n) = \{v \in e' : I(v) \leq sn\}$ (cf. (6)) has at least $z = \lfloor 4k^2 r / \ln r \rfloor$ elements. More shortly,

$$(9) \quad X(T) = \bigcup_{e' \in E(T)} \{|M(e', n)| \geq z\}.$$

Let $\overline{X(T)}$ denote the complement of $X(T)$. Suppose that the event $\widetilde{W}(e, \alpha, T, l) \cap \overline{X(T)}$ occurs.

When a sub- r -tree T with root e is given, it automatically defines the edges e_1, \dots, e_q of T that share a vertex with e . It also defines for every $j \in [q]$, the vertex $\{v_j\} = e \cap e_j$ and the number y_j of edges in $T(e_j)$. Below, when T is given, we will denote $Q = Q(T, e) \doteq \{v_1, \dots, v_q\}$. Every outcome of Evolution such that $\widetilde{W}(e, \alpha, T, l) \cap \overline{X(T)}$ occurs defines the vector (l_1, \dots, l_q) such that $s(l_j - 1) < I(v_j) \leq sl_j$ for each $j \in [q]$. By $(\widetilde{W}3)$, for each $j \in [q]$, $l_j \in [l]$. By $(\widetilde{W}4)$ and $(\widetilde{W}2)$, all l_j are distinct.

Let $\Theta_0 = \Theta_0(q, l)$ be the set of vectors (l_1, \dots, l_q) such that

$$(10) \quad \text{all } l_1, \dots, l_q \text{ are distinct and belong to } [l].$$

By the previous paragraph,

$$(11) \quad \widetilde{W}(e, \alpha, T, l) \cap \overline{X(T)} = \widetilde{W}(e, \alpha, T, l) \cap \overline{X(T)} \cap \{(l_1, \dots, l_q) \in \Theta_0\}.$$

Let $\Theta_1(q, l) = \Theta_0(q, l-1)$, i.e. the set of $(l_1, \dots, l_q) \in \Theta_0$ such that $l_j \leq l-1$ for all $j \in [q]$. Let $\Theta_2 = \Theta_2(q, l) = \Theta_0(q, l) - \Theta_1(q, l)$. For $i = 1, 2$, let

$$F_i(e, \alpha, T, l) = \widetilde{W}(e, \alpha, T, l) \cap \overline{X(T)} \cap \{(l_1, \dots, l_q) \in \Theta_i\}.$$

By (11),

$$(12) \quad \widetilde{W}(e, \alpha, T, l) \subseteq X(T) \cup F_1(e, \alpha, T, l) \cup F_2(e, \alpha, T, l).$$

Our goal is to prove that for $i = 1, 2$,

$$(13) \quad \mathbf{Pr}(F_i(e, \alpha, T, l)) \leq 0.4\varepsilon D^{-y} \left(\frac{r}{\ln r}\right)^{\frac{n-l}{n+1}}.$$

Since by Lemma 2 and (8), $\mathbf{Pr}(X(T)) \leq k\varepsilon^{0.5r} < 0.1\varepsilon D^{-y}$, (12) and (13) will imply the lemma.

Observe that the condition “ $l_j \leq l-1$ for all $j \in [q]$ ” in the definition of $\Theta_1(q, l)$ implies that if $\widetilde{W}(e, \alpha, T, l)$ occurs, then all conditions $(\widetilde{W}1)$ – $(\widetilde{W}8)$ are satisfied for the event $\widetilde{W}(e, \alpha, T, l-1)$. Thus,

$$(14) \quad F_1(e, \alpha, T, l) \subseteq \widetilde{W}(e, \alpha, T, l-1).$$

By the induction assumption,

$$(15) \quad \mathbf{Pr}(\widetilde{W}(e, \alpha, T, l-1)) \leq \varepsilon D^{-y} \left(\frac{r}{\ln r}\right)^{(n-l+1)/(n+1)}.$$

Let $Z(e, l)$ be the event that $I(v) > sl$ for each $v \in e - M(e, l - 1)$. If $\widetilde{W}(e, \alpha, T, l) \cap \overline{X(T)}$ holds, then by ($\widetilde{W}8$), $Z(e, l)$ occurs. Thus

$$(16) \quad F_1(e, \alpha, T, l) \subseteq Z(e, l).$$

Since we already know whether the event $\widetilde{W}(e, \alpha, T, l - 1)$ occurs or not after Step $s(l - 1)$, for each $v \in e - M(e, l - 1)$,

$$\begin{aligned} \Pr(\{I(v) > sl \mid \widetilde{W}(e, \alpha, T, l - 1)\}) &\leq \frac{\Pr(I(v) > sl)}{\Pr(I(v) > s(l - 1))} \\ &= \frac{1 - lps}{1 - (l - 1)ps}. \end{aligned}$$

Since all random variables $I(v)$ are mutually independent,

$$(17) \quad \Pr(\{Z(e, l) \mid \widetilde{W}(e, \alpha, T, l - 1)\}) \leq \left(\frac{1 - lps}{1 - (l - 1)ps}\right)^{r - |M(e, l - 1)|} \\ \leq (1 - ps)^{r - |M(e, l - 1)|}.$$

Therefore,

$$\begin{aligned} &\Pr(\{Z(e, l) \mid \widetilde{W}(e, \alpha, T, l - 1)\}) \\ &\leq \sum_{M \subseteq e} \Pr\{M = M(e, l - 1)\} (1 - ps)^{r - |M|}. \end{aligned}$$

By Lemma 2, $\Pr(|M(e, l - 1)| \geq z) \leq \varepsilon^{0.5r}$. Hence

$$\begin{aligned} &\sum_{M \subseteq e} \Pr\{M = M(e, l - 1)\} (1 - ps)^{r - |M|} \\ &\leq \varepsilon^{0.5r} + \sum_{M \subseteq e: |M| < z} \Pr\{M = M(e, l - 1)\} (1 - ps)^{r - |M|} \\ &\leq \varepsilon^{0.5r} + (1 - ps)^{r - z} \leq \varepsilon^{0.5r} + \exp\left\{-psr \left(1 - \frac{4k^2}{\ln r}\right)\right\}. \end{aligned}$$

Since $ps \geq \frac{\ln r}{(n+1)r} - p$, by the definition of p and s ,

$$\begin{aligned} \exp\left\{-psr \left(1 - \frac{4k^2}{\ln r}\right)\right\} &\leq \exp\left\{-\left(\frac{\ln r}{n+1} - pr\right) \left(1 - \frac{4k^2}{\ln r}\right)\right\} \\ &\leq \exp\left\{-\frac{\ln r}{n+1} + \frac{4k^2}{n+1} + pr\right\}. \end{aligned}$$

Recall that $4k^2 = -\ln \varepsilon$. Since $p < 2^{-k^r r}$, $pr < \frac{4k^2}{n+1}$ and hence

$$\exp \left\{ -\frac{\ln r}{n+1} + \frac{4k^2}{n+1} + pr \right\} \leq r^{\frac{-1}{n+1}} e^{8k^2/(n+1)} \leq \frac{1}{\varepsilon} r^{\frac{-1}{n+1}}.$$

By (3), $\ln r \geq 2\varepsilon^{-2(n+1)}$ and by (4), $\varepsilon = \exp\{-4k^2\}$. So,

$$(18) \quad \frac{1}{\varepsilon} r^{\frac{-1}{n+1}} \leq \frac{1}{\varepsilon} \left(\frac{r}{\ln r} \right)^{\frac{-1}{n+1}} \varepsilon^2 = \left(\frac{r}{\ln r} \right)^{\frac{-1}{n+1}} \exp\{-4k^2\} < 0.1 \left(\frac{r}{\ln r} \right)^{\frac{-1}{n+1}}.$$

By (14) and (16),

$$\Pr(F_1(e, \alpha, T, l)) \leq \Pr(Z(e, l) \cap \widetilde{W}(e, \alpha, T, l-1)).$$

Thus by (15) and (18),

$$\begin{aligned} & \Pr(F_1(e, \alpha, T, l)) \\ & \leq \Pr(\{Z(e, l) \mid \widetilde{W}(e, \alpha, T, l-1)\}) \Pr(\widetilde{W}(e, \alpha, T, l-1)) \\ & \leq \left(\varepsilon^{0.5r} + 0.1 \left(\frac{r}{\ln r} \right)^{\frac{-1}{n+1}} \right) \Pr(\widetilde{W}(e, \alpha, T, l-1)) \\ & \leq \varepsilon^{0.5r} + 0.1\varepsilon D^{-y} \left(\frac{r}{\ln r} \right)^{\frac{n-l}{n+1}}. \end{aligned}$$

Since by (8), $\varepsilon^{0.5r-1} < 0.03D^{-k} \leq 0.03D^{-y}$, this implies (13) for $i = 1$.

Now we will prove (13) for $i = 2$. Suppose that $F_2(e, \alpha, T, l)$ occurs.

- Then, by definition, $\vec{l} = (l_1, \dots, l_q) \in \Theta_2 = \Theta_2(q, l)$. In particular, there exists $j^* \in [q]$ such that $l_{j^*} = l$.

- By $(\widetilde{W}5)$, for every $j \in [q]$, the event $\widetilde{W}(e_j, \alpha - 2^{l_j-1}, T(e_j), l_j - 1)$ occurs.

- Also, for every $j \in [q]$, there exists $h_j \in [s]$ such that $I(v_j) = s(l_j - 1) + h_j$.

- By $(\widetilde{W}6)$, for every $j \in [q]$, for every $u \in e_j$ with $I(u) > (l_j - 1)s$, we have also $I(u) \geq (l_j - 1)s + h_j$.

- By $(\widetilde{W}1)$ and $(\widetilde{W}2)$, coloring $\phi(e)$ belongs to the set $\Psi(l_1, \dots, l_q)$ of colorings ψ of e such that the following two properties hold:

(P1) $\psi(v_j) = \alpha - 2^{l_j-1}$ for all $j \in [q]$.

(P2) $\psi(v) \in \{\psi(v_1), \dots, \psi(v_q), \alpha\}$ for all $v \in e$.

For $j \in [q]$, let $A_j = \{v \in e : \phi(v) = \phi(v_j)\}$ and $a_j = |A_j| - 1$.

• By $(\widetilde{W}7)$, for every $j \in [q]$, for each $u \in A_j - v_j$, $(l_j - 1)s + 1 \leq I(u) \leq (l_j - 1)s + h_j$.

Thus, in order for $F_2(e, \alpha, T, l)$ to occur, all of the following events should occur:

(R1) There exists a vector $\vec{\ell} = (l_1, \dots, l_q) \in \Theta_2 = \Theta_2(q, l)$ such that for every $j \in [q]$, the event $R_1(j, l_j) := \widetilde{W}(e_j, \alpha - 2^{l_j-1}, T(e_j), l_j - 1)$ occurs. Let $R_1(\vec{\ell}) := \bigcap_{j=1}^q R_1(j, l_j)$.

(R2) For this vector $\vec{\ell} = (l_1, \dots, l_q)$, there exists a vector $\vec{h} = (h_1, \dots, h_q) \in [s]^q$ such that for every $j \in [q]$, the event $R_2(j, l_j, h_j) := \{I(v_j) = s(l_j - 1) + h_j\}$ occurs. Let $R_2(\vec{\ell}, \vec{h}) := \bigcap_{j=1}^q R_2(j, l_j, h_j)$.

(R3) For these vectors $\vec{\ell} = (l_1, \dots, l_q)$ and $\vec{h} = (h_1, \dots, h_q)$, for every $j \in [q]$ and every $u \in e_j$, the event

$$R_3(u, j, l_j, h_j) := \{I(u) \notin [s(l_j - 1) + 1, s(l_j - 1) + h_j - 1]\}$$

occurs. Let $R_3(j, l_j, h_j) = \bigcap_{u \in e_j - v_j} R_3(u, j, l_j, h_j)$ and $R_3(\vec{\ell}, \vec{h}) := \bigcap_{j=1}^q R_3(j, l_j, h_j)$.

(R4) For this vector $\vec{\ell} = (l_1, \dots, l_q)$, there exists a coloring $\psi \in \Psi(\vec{\ell})$ such that the event $R_4(\phi, \vec{\ell}) := \{\phi(e) = \psi\}$ occurs.

(R5) For these $\vec{\ell}$, \vec{h} , and ψ , if for $j \in [q]$, we define $A_j = A_j(\psi) := \{v \in e : \psi(v) = \psi(v_j)\}$, then for every $j \in [q]$ and $u \in A_j$, the event

$$R_5(\psi, u, j, l_j, h_j) := \{(l_j - 1)s + 1 \leq I(u) \leq (l_j - 1)s + h_j\}$$

occurs. Let $R_5(\psi, \vec{\ell}, \vec{h}) := \bigcap_{j=1}^q \bigcap_{u \in A_j - v_j} R_5(\psi, u, j, l_j, h_j)$.

We conclude that

$$F_2(e, \alpha, T, l) \subseteq \bigcup_{\vec{\ell} \in \Theta_2} R_1(\vec{\ell}) \cap \left[\bigcup_{\vec{h} \in [s]^q} \left(R_2(\vec{\ell}, \vec{h}) \cap R_3(\vec{\ell}, \vec{h}) \right) \right. \\ \left. \cap \bigcup_{\psi \in \Psi(\vec{\ell})} \left(R_4(\psi, \vec{\ell}) \cap R_5(\psi, \vec{\ell}, \vec{h}) \right) \right].$$

So, we estimate

$$\begin{aligned}
(19) \quad & \Pr(F_2(e, \alpha, T, l)) \\
& \leq \sum_{\vec{\ell} \in \Theta_2} \Pr(R_1(\vec{\ell})) \sum_{\vec{h} \in [s]^q} \Pr(\{R_2(\vec{\ell}, \vec{h}) \cap R_3(\vec{\ell}, \vec{h}) \mid R_1(\vec{\ell})\}) \\
& \quad \times \sum_{\psi \in \Psi(\vec{\ell})} \Pr(\{R_4(\psi, \vec{\ell}) \mid R_1(\vec{\ell}) \cap R_2(\vec{\ell}, \vec{h}) \cap R_3(\vec{\ell}, \vec{h})\}) \\
& \quad \times \Pr(\{R_5(\psi, \vec{\ell}, \vec{h}) \mid R_1(\vec{\ell}) \cap R_2(\vec{\ell}, \vec{h}) \cap R_3(\vec{\ell}, \vec{h}) \cap R_4(\psi, \vec{\ell})\}).
\end{aligned}$$

We now will gradually evaluate and simplify the expression in (19). We start from $R_1(\vec{\ell})$. Since the vertex sets of $T(e_j)$ for distinct j are disjoint and by Lemma 10, for every j the event $R_1(j, l_j) = \widetilde{W}(e_j, \alpha - 2^{l_j-1}, T(e_j), l_j - 1)$ depends only on the values of $I(v)$ and $\phi(v)$ for $v \in V(T(e_j))$,

$$\Pr(R_1(\vec{\ell})) = \prod_{j=1}^q \Pr(R_1(j, l_j)).$$

If $T(e_j)$ has y_j edges, then by the induction assumption,

$$\Pr(R_1(j, l_j)) \leq \varepsilon D^{-y_j} \left(\frac{r}{\ln r}\right)^{(n-l_j+1)/(n+1)}.$$

Thus,

$$(20) \quad \Pr(R_1(\vec{\ell})) \leq \prod_{j=1}^q \varepsilon D^{-y_j} \left(\frac{r}{\ln r}\right)^{(n-l_j+1)/(n+1)}.$$

Now, consider the events $R_3(u, j, l_j, h_j)$. Since all $I(v)$ are independent and all trees T_j are vertex-disjoint, $R_3(u, j, l_j, h_j)$ does not depend on $R_1(j', l_{j'})$ for all $j' \neq j$. Furthermore, if $I(u) > s(l_j - 1)$, then $\widetilde{W}(e_j, \alpha - 2^{l_j-1}, T(e_j), l_j - 1)$ does not depend on the particular value of $I(u)$ in this range. So,

$$\Pr(\{R_3(u, j, l_j, h_j) \mid R_1(\vec{\ell})\}) \leq \frac{1 - p(l_j - 1)s - p(h_j - 1)}{1 - p(l_j - 1)s} \leq 1 - p(h_j - 1).$$

By the independence of $I(u)$ for distinct u , similarly to (17) and the argument following (17), we have

$$\begin{aligned}
& \Pr \left(\{ R_3(j, l_j, h_j) \mid R_1(\vec{\ell}) \} \right) \\
& \leq \sum_{M \subseteq e_j} \Pr \{ M = M(e_j, l_j - 1) \} (1 - p(h_j - 1))^{r-|M|-1} \leq \varepsilon^{0.5r} \\
& \quad + \sum_{M \subseteq e: |M| < z} \Pr \{ M = M(h_j - 1) \} (1 - p(h_j - 1))^{r-|M|-1} \\
& \leq \varepsilon^{0.5r} + (1 - p(h_j - 1))^{r-z}.
\end{aligned}$$

Since $\varepsilon^{0.5r} \leq 0.1(1 - ps)^r \leq 0.1(1 - p(h_j - 1))^{r-z}$, again by the independence of $I(u)$ for distinct u , we conclude that

$$(21) \quad \Pr \left(\{ R_3(\vec{\ell}, \vec{h}) \mid R_1(\vec{\ell}) \} \right) \leq \prod_{j=1}^q 1.1(1 - p(h_j - 1))^{r-z}.$$

Now, we consider events $R_2(j, l_j, h_j) = \{ I(v_j) = s(l_j - 1) + h_j \}$. Let $j \in [q]$. Similarly to $R_3(u, j, l_j, h_j)$, $R_2(j, l_j, h_j)$ does not depend on $R_1(j', l_{j'})$ for all $j' \neq j$, and

$$\Pr \left(\{ R_2(j, l_j, h_j) \mid R_1(\vec{\ell}) \} \right) \leq \frac{p}{1 - ps(l_j - 1)} \leq p \left(1 + \frac{\ln r}{r} \right).$$

Moreover, again by the independence of $I(u)$ for distinct u , $R_2(j, l_j, h_j)$ is independent of all other $R_2(j', l_{j'}, h_{j'})$ and of $R_3(\vec{\ell}, \vec{h})$. Thus, since $q \leq k \ll \ln r$,

$$(22) \quad \Pr \left(\{ R_2(\vec{\ell}, \vec{h}) \mid R_1(\vec{\ell}) \cap R_3(\vec{\ell}, \vec{h}) \} \right) \leq p^q \left(1 + \frac{\ln r}{r} \right)^q \leq 2p^q,$$

and together with (21), we obtain

$$(23) \quad \Pr \left(\{ R_2(\vec{\ell}, \vec{h}) \cap R_3(\vec{\ell}, \vec{h}) \mid R_1(\vec{\ell}) \} \right) \leq 2p^q \prod_{j=1}^q 1.1(1 - p(h_j - 1))^{r-z}.$$

Now for given $\vec{\ell}, \vec{h}$, and $\psi \in \Psi(\vec{\ell})$, we evaluate

$$\tilde{P}_4(\psi, \vec{\ell}, \vec{h}) := \Pr \left(\{ R_4(\psi, \vec{\ell}) \mid R_1(\vec{\ell}) \cap R_2(\vec{\ell}, \vec{h}) \cap R_3(\vec{\ell}, \vec{h}) \} \right)$$

Observe that each event $R_1(\vec{\ell}) \cap R_2(\vec{\ell}, \vec{h}) \cap R_3(\vec{\ell}, \vec{h})$ already fixes the colors of v_1, \dots, v_q in ϕ , but all other vertices of e are “free”. Since for each $v \in e$, the value of $\phi(v)$ is chosen independently of all other vertices, for each $\vec{\ell}, \vec{h}$, and $\psi \in \Psi(\vec{\ell})$, we have

$$(24) \quad \tilde{P}_4(\psi, \vec{\ell}, \vec{h}) \leq k^{q-r}.$$

Finally, for given $\vec{\ell}, \vec{h}$, and $\psi \in \Psi(\vec{\ell})$, consider the event

$$\tilde{R}_5(\psi, \vec{\ell}, \vec{h}) := \{R_5(\psi, \vec{\ell}, \vec{h}) \mid R_1(\vec{\ell}) \cap R_2(\vec{\ell}, \vec{h}) \cap R_3(\vec{\ell}, \vec{h}) \cap R_4(\psi, \vec{\ell})\}.$$

For $j \in [q]$, let let $A_j = A_j(\psi) := \{v \in e : \psi(v) = \psi(v_j)\}$ and $a_j = |A_j| - 1$. By definition, for every $j \in [q]$ and every $u \in A_j - v_j$, the probability of the event

$$R_5(\psi, u, j, l_j, h_j) := \{(l_j - 1)s + 1 \leq I(u) \leq (l_j - 1)s + h_j\}$$

is at most ph_j . Since the values of $I(v)$ for $v \in e - \{v_1, \dots, v_q\}$ do not depend on $R_1(\vec{\ell}) \cap R_2(\vec{\ell}, \vec{h}) \cap R_3(\vec{\ell}, \vec{h}) \cap R_4(\psi, \vec{\ell})$, and are independent of each other for distinct v ,

$$\Pr(\tilde{R}_5(\psi, \vec{\ell}, \vec{h})) \leq \prod_{j=1}^q (ph_j)^{a_j}.$$

Since each $\psi \in \Psi(l_1, \dots, l_q)$ is completely defined when we choose disjoint sets $A_1 - v_1, \dots, A_q - v_q$ in $e - \{v_1, \dots, v_q\}$,

$$(25) \quad \sum_{\psi \in \Psi(\vec{\ell})} \tilde{P}_4(\psi, \vec{\ell}, \vec{h}) \Pr(\tilde{R}_5(\psi, \vec{\ell}, \vec{h})) \\ \leq \sum_{a_1=0}^r \sum_{a_2=0}^r \cdots \sum_{a_q=0}^r \binom{r}{a_1} \binom{r}{a_2} \cdots \binom{r}{a_q} k^{q-r} \prod_{j=1}^q (ph_j)^{a_j}.$$

Thus plugging (20), (23), and (25) into (19), we have

$$(26) \quad \Pr(F_2(e, \alpha, T, l)) \leq \sum_{\vec{\ell} \in \Theta_2} \left(\prod_{j=1}^q \varepsilon D^{-y_j} \left(\frac{r}{\ln r} \right)^{(n-l_j+1)/(n+1)} \right)$$

$$(27) \quad \times \sum_{\vec{h} \in [s]^q} 2p^q \left(\prod_{j=1}^q 1.1(1 - p(h_j - 1))^{r-z} \right) \\ \times \sum_{a_1=0}^r \sum_{a_2=0}^r \cdots \sum_{a_q=0}^r \binom{r}{a_1} \binom{r}{a_2} \cdots \binom{r}{a_q} k^{q-r} (ph_j)^{a_j}.$$

We now will simplify and estimate the expression in (27). First observe that

$$(1 - p(h_j - 1))^{r-z} \leq (1 - p)^{(r-z)(h_j-1)}.$$

Thus since $0 < p < 2^{-k^r r}$ and $h_j \leq s \leq \frac{\ln r}{p(n+1)r}$, we have

$$1.1(1 - p(h_j - 1))^{r-z} \leq 1.1(1 - p)^{(r-z)(h_j-1)} \leq 1.2(1 - p)^{(r-z)h_j}.$$

So, the expression in (27) is at most

$$\begin{aligned} & 2 \sum_{\vec{h} \in [s]^q} p^q \left(\prod_{j=1}^q 1.2(1 - p)^{(r-z)h_j} \right) \\ & \times \sum_{a_1=0}^r \sum_{a_2=0}^r \cdots \sum_{a_q=0}^r \binom{r}{a_1} \binom{r}{a_2} \cdots \binom{r}{a_q} k^{q-r} (ph_j)^{a_j} \\ (28) \quad & = 2k^{-r} \sum_{h_1=1}^s \sum_{a_1=0}^r 1.2p(1 - p)^{(r-z)h_1} \binom{r}{a_1} k(ph_1)^{a_1} \\ & \times \left[\sum_{h_2=1}^s \sum_{a_2=0}^r 1.2p(1 - p)^{(r-z)h_2} \binom{r}{a_2} k(ph_2)^{a_2} \right. \\ (29) \quad & \times \left. \left[\cdots \sum_{h_q=1}^s \sum_{a_q=0}^r 1.2p(1 - p)^{(r-z)h_q} \binom{r}{a_q} k(ph_q)^{a_q} \right] \cdots \right]. \end{aligned}$$

For $j = q, q-1, \dots, 1$ (in this order), we can estimate

$$\begin{aligned} & \sum_{h_j=1}^s \sum_{a_j=0}^r 1.2p(1 - p)^{(r-z)h_j} \binom{r}{a_j} k(ph_j)^{a_j} \\ & \leq 1.2pk \sum_{h_j=1}^s (1 - p)^{rh_j - zs} \sum_{a_j=0}^r \binom{r}{a_j} (ph_j)^{a_j} \\ & \leq 1.2pk(1 - p)^{-zs} \sum_{h_j=1}^s (1 - p)^{rh_j} (1 + ph_j)^r \\ & \leq 1.2pke^{\frac{pzs}{1-p}} \sum_{h_j=1}^s (1 - p)^{rh_j} (1 + ph_j)^r \end{aligned}$$

$$\begin{aligned} &\leq 1.2pk e^{\frac{z \ln r}{(1-p)(n+1)^r}} \sum_{h_j=1}^s (1-p)^{rh_j} (1+p)^{rh_j} \\ &< 1.2(ps)k e^{\frac{z \ln r}{(1-p)(n+1)^r}} \leq 1.2(ps)k e^{\frac{4k^2}{(1-p)(n+1)}}. \end{aligned}$$

Since $ps \leq \frac{\ln r}{r(n+1)}$, $n+1 \geq 2$, and $4k^2 = -\ln \varepsilon$, we have

$$1.2(ps)k e^{\frac{4k^2}{(1-p)(n+1)}} \leq 1.2 \frac{\ln r}{r(n+1)} k \varepsilon^{-1/(1-p)(n+1)} < e^{-3k^2/2} \frac{k \ln r}{2r(n+1)\varepsilon}.$$

Plugging these bounds into (28)–(29), we obtain that the expression in (27) does not exceed

$$2k^{-r} \prod_{j=1}^q \left(e^{-3k^2/2} \frac{k \ln r}{2r(n+1)\varepsilon} \right) \leq k^{-r} \prod_{j=1}^q \left(e^{-3k^2/2} \frac{k \ln r}{r(n+1)\varepsilon} \right).$$

Thus, by (26)–(27)

$$\begin{aligned} (30) \quad &\Pr(F_2(e, \alpha, T, l)) \\ &\leq \sum_{\vec{\ell} \in \Theta_2} k^{-r} \prod_{j=1}^q \left(\varepsilon D^{-y_j} \left(\frac{r}{\ln r} \right)^{\frac{n-l_j+1}{n+1}} \right) \left(e^{-3k^2/2} \frac{k \ln r}{r(n+1)\varepsilon} \right). \end{aligned}$$

Note that

$$(31) \quad k^{-r} D^{-y_1 \cdots y_q} = k^{-r} D^{-y+1} \leq D^{-y} \varepsilon \left(\frac{r}{\ln r} \right)^{\frac{n}{n+1}}.$$

Hence (recalling that $D = \varepsilon k^r \left(\frac{r}{\ln r} \right)^{\frac{n}{n+1}}$), we may rewrite (30) as

$$(32) \quad \Pr(F_2(e, \alpha, T, l)) \leq D^{-y} \varepsilon \left(\frac{r}{\ln r} \right)^{\frac{n}{n+1}} \sum_{\vec{\ell} \in \Theta_2} \prod_{j=1}^q \left(\varepsilon \left(\frac{r}{\ln r} \right)^{\frac{n-l_j+1}{n+1}} \right) \left(e^{-3k^2/2} \frac{k \ln r}{r(n+1)\varepsilon} \right).$$

By the definition of Θ_2 , for every $\vec{\ell} \in \Theta_2$ there is j^* such that $l_{j^*} = l$. For every $j \neq j^*$, we estimate

$$(33) \quad \varepsilon \left(\frac{r}{\ln r} \right)^{\frac{n-l_j+1}{n+1}} e^{-3k^2/2} \frac{k \ln r}{r(n+1)\varepsilon} \leq \frac{k}{n+1} e^{-3k^2/2} < \frac{1}{n+1}.$$

For $j = j^*$ we will gain more, since $l_{j^*} = l$. Namely,

$$(34) \quad \varepsilon \left(\frac{r}{\ln r} \right)^{\frac{n-l+1}{n+1}} e^{-3k^2/2} \frac{k \ln r}{r(n+1)\varepsilon} \leq e^{-3k^2/2} \frac{k}{n+1} \left(\frac{\ln r}{r} \right)^{\frac{l}{n+1}}.$$

Plugging (33) and (34) into (32), we have

$$\begin{aligned} & \Pr(F_2(e, \alpha, T, l)) \\ & \leq D^{-y} \varepsilon \left(\frac{r}{\ln r} \right)^{\frac{n}{n+1}} \sum_{\vec{\ell} \in \Theta_2} (n+1)^{-q+1} \left(e^{-3k^2/2} \frac{k}{n+1} \left(\frac{\ln r}{r} \right)^{\frac{l}{n+1}} \right). \end{aligned}$$

Since the summands in the last expression do not depend on the choice of $\vec{\ell} \in \Theta_2$ and $|\Theta_2| \leq (l+1)^q \leq (n+1)^q$, we have

$$\begin{aligned} & \Pr(F_2(e, \alpha, T, l)) \\ & \leq D^{-y} \varepsilon \left(\frac{r}{\ln r} \right)^{\frac{n}{n+1}} \frac{|\Theta_2|}{(n+1)^{q-1}} \left(e^{-3k^2/2} \frac{k}{n+1} \left(\frac{\ln r}{r} \right)^{\frac{l}{n+1}} \right) \leq \\ & \leq D^{-y} \varepsilon \left(\frac{r}{\ln r} \right)^{\frac{n-l}{n+1}} e^{-3k^2/2} k \leq 0.4 \varepsilon D^{-y} \left(\frac{r}{\ln r} \right)^{\frac{n-l}{n+1}}. \end{aligned}$$

This proves (13) for $i = 2$ and thus the lemma. ■

Applying Lemma 11 for $l = n$, we get the following immediate consequence.

Corollary 1. *Let $e \in E(G)$ and $\alpha \in [k]$. Let $D := \varepsilon k^r \left(\frac{r}{\ln r} \right)^{\frac{n}{n+1}}$. Let T be a rooted sub- r -tree of G with root e . If T has y edges, then*

$$\Pr(\widetilde{W}(e, \alpha, T, n)) \leq \varepsilon D^{-y}.$$

6. PROOF OF THEOREM 2

Recall the following version of the Local Lemma.

Theorem 4 [1]. *Let A_1, A_2, \dots, A_N be any events. Let S_1, S_2, \dots, S_N be subsets of $[n]$ such that for each i , A_i is independent of the events $\{A_j : j \in ([N] - S_i)\}$. If there exist numbers $x_1, x_2, \dots, x_N \in [0, 1)$ such that for all $i \in [N]$, $\Pr[A_i] \leq x_i \prod_{j \in S_i} (1 - x_j)$, Then,*

$$\Pr \left[\bigwedge_{i \in [N]} \overline{A_i} \right] \geq \prod_{i \in [N]} (1 - x_i) > 0.$$

Radhakrishnan and Srinivasan used it in the following form.

Lemma 12 [6]. *Let A_1, A_2, \dots, A_N be any events. Let S_1, S_2, \dots, S_N be subsets of $[N]$ such that for each i , A_i is independent of the events $\{A_j : j \in ([N] - S_i)\}$. If for all $i \in [N]$, $\Pr(A_i) < \frac{1}{2}$ and $\sum_{j \in S_i} \Pr(A_j) \leq \frac{1}{4}$, then*

$$\Pr \left[\bigwedge_{i \in [N]} \overline{A_i} \right] > 0.$$

Proof. We show that if the conditions of this lemma hold, then the conditions of Theorem 4 hold for $x_i = 2 \Pr(A_i)$, $i \in [N]$. Indeed, with so defined x_i , inequality

$$\Pr[A_i] \leq x_i \prod_{j \in S_i} (1 - x_j)$$

follows if $\prod_{j \in S_i} (1 - x_j) \geq \frac{1}{2}$ holds. Furthermore,

$$\prod_{j \in S_i} (1 - x_j) \geq 1 - \sum_{j \in S_i} x_j = 1 - 2 \sum_{j \in S_i} \Pr(A_j) \geq \frac{1}{2} \left(\text{since } \sum_{j \in S_i} \Pr(A_j) \leq \frac{1}{4} \right).$$

Hence by Theorem 4, we have the result. ■

Lemma 13. *Let $0 < \varepsilon \leq 4^{-k} k^{-4}$. If $\Pr(\widetilde{W}(e, \alpha, T, n)) \leq \varepsilon D^{-y}$ for every $\alpha \in [k]$, every sub- r -tree T of G with $y \leq k$ edges and for every $e \in E(T)$, then with positive probability, none of these events occurs.*

Proof. Consider the probability space of the outcomes of Evolution. Let the events A_1, \dots, A_N be the events $\widetilde{W}(e, \alpha, T, n)$ for all $e \in E(G)$, all $\alpha \in [k]$ and all sub- r -trees T of G containing e with at most k edges. It is enough to verify that the conditions of Lemma 12 hold for our events A_1, \dots, A_N . Each of the conditions $\Pr(\widetilde{W}(e, \alpha, T, n)) < 1/2$ immediately follows from Corollary 1. By Lemma 10, for the event $A_i = \widetilde{W}(e, \alpha, T, n)$, we can take S_i equal to the set of all events $\widetilde{W}(e', \alpha', T', n)$ such that $V(T') \cap V(T) \neq \emptyset$.

Now, fix an event $A_i = \widetilde{W}(e, \alpha, T, n)$, where T has y edges, and estimate $\sum_{j \in S_i} \Pr(A_j)$. Let $\widetilde{W}(e', \alpha', T', n) \in S_i$ and suppose that the size of T' is y' . Then some edge e'' of T' intersects $V(T)$ (in particular, e'' can be an edge of T , too). The number of ways to choose an edge that intersects $V(T)$ is at most $D + 1$ if $y = 1$, and is at most yD , if $y > 1$. In any case, this number is not greater than kD . By Lemma 7, G contains at most $(4D)^{y'-1}$ r -trees of size y' containing edge e'' . In each of such trees, there are y' ways to choose a root, e' , and k ways to choose the color α' . Since $\Pr(\widetilde{W}(e', \alpha', T', n)) \leq \varepsilon D^{-y'}$, it follows that

$$(35) \quad \sum_{j \in S_i} \Pr(A_j) \leq \sum_{y'=1}^k (kD) (4D)^{y'-1} y' k (\varepsilon D^{-y'}) = \sum_{y'=1}^k k^2 y' 4^{y'-1} \varepsilon \leq k^4 4^{k-1} \varepsilon.$$

Since $0 < \varepsilon \leq 4^{-k} k^{-4}$, the last expression in (35) is at most $1/4$. Thus we are done by Lemma 12. ■

Now we are ready to complete the proof of the main theorem. Indeed, let G be a hypergraph satisfying the conditions of the theorem. Consider procedure Evolution. By Corollary 1, for each y -edge r -tree T , each edge $e \in E(T)$ and each $\alpha \in [k]$, $\Pr(\widetilde{W}(e, \alpha, T, n)) \leq \varepsilon D^{-y}$. For $k \geq 2$, we have $\varepsilon = \exp\{-4k^2\} < 4^{-k} k^{-4}$. So, by Lemma 13, with positive probability none of the events $\widetilde{W}(e, \alpha, T, n)$ occurs. It follows that in some outcome of Evolution none of the events $\widetilde{W}(e, \alpha, T, n)$ occurs. By Lemma 9, in this outcome none of the events $W(e, \alpha, T, n)$ occurs. But then the resulting k -coloring will be proper.

REFERENCES

- [1] N. Alon and J. Spencer, *The Probabilistic Method*, Second Edition, Wiley (2000).
- [2] P. Erdős and L. Lovász, Problems and results on 3-chromatic hypergraphs and some related questions, in: *Infinite and Finite Sets*, A. Hajnal et al., editors, Colloq. Math. Soc. J. Bolyai **11**, North Holland (Amsterdam, 1975), pp. 609–627.
- [3] A. V. Kostochka, Coloring uniform hypergraphs with few colors, *Random Structures and Algorithms*, **24** (2004), 1–10.
- [4] A. Kostochka and M. Kumbhat, Coloring simple uniform hypergraphs with few edges, to appear in *Random Structures and Algorithms*.
- [5] A. Kostochka and V. Rödl, Constructions of sparse uniform hypergraphs with high chromatic number, submitted.
- [6] J. Radhakrishnan and A. Srinivasan, Improved bounds and algorithms for hypergraph two-coloring, *Random Structures and Algorithms*, **16** (2000), 4–32.

Alexandr V. Kostochka

*University of Illinois,
Urbana, IL, 61801*

and

*Institute of Mathematics,
Novosibirsk,
630090, Russia*

e-mail: kostochk@math.uiuc.edu

Mohit Kumbhat

*Department of Mathematics,
University of Illinois,
Urbana, IL, 61801,
U.S.A.*

e-mail: kumbhat2@uiuc.edu

Vojtěch Rödl

*Department of Mathematics and
Computer Science,*

*Emory University,
Atlanta, GA 30322,
U.S.A.*

e-mail: rodl@mathcs.emory.edu