WHAT IS MATH RESEARCH?

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What is math research?

I'm a mathematician. I spend a large portion of each working day "doing research." To those outside the arcana of academic mathematics, this can be an opaque phrase. My utterance of "doing research" is almost always met with "What is math research?" or "How do you research math?" I have found it surprising how often this pattern has played out. The persistence and consistency of these questions persuaded me that a suitable answer should be given.

Before we proceed further, let me note that, by the time you are done reading, you probably still won't have any idea what a research mathematician does. It's the nature of the beast, unfortunately. It is my hope, however, that by reading this note you will come to see mathematicians less as wizards or purveyors of esoteric secrets, and more as explorers of beautiful but untamed wilderness.

Necessarily I will speak of math, and therefore this note assumes a certain level of mathematics knowledge. I have tried to keep the presentation simple, but if you find the going too tough, read the first few lines of the section titled "What is research?" for the TL/DR version.

What is math?

In order to understand *math research*, it is important to have some notion of what we mean by *math*. As we are wont to do in the Information Age, let us turn to Wikipedia for some insight.

Mathematics*...is the study of topics such as quantity (numbers), structure, space, and change. There is a range of views among mathematicians and philosophers as to the exact scope and definition of mathematics.*

Mathematicians seek out patterns and use them to formulate new conjectures. Mathematicians resolve the truth or falsity of conjectures by mathematical proof. When mathematical structures are good models of real phenomena, then mathematical reasoning can provide insight or predictions about nature. Through the use of abstraction and logic, mathematics developed from counting, calculation, measurement, and the systematic study of the shapes and motions of physical objects. Practical mathematics has been a human activity from as far back as written records exist. The research required to solve mathematical problems can take years or even centuries of sustained inquiry.

I highly recommend reading the entire Wikipedia page "Mathematics."

The most important thing to know about math is that it deals with constructs of pure logic and abstraction. The mathematician does not deal in the actual and the earthy, like a biologist with his frogs or a physicist with her particles. Of course, biologists and physicists use the tools of mathematics, but a mathematician studies structures and objects that are independent of any potential instantiations in reality.

If this is already getting too heady, let's stop here for an example. You probably remember learning about Euclidean or plane geometry in high school (this would fall under what the Wikipedia article calls "space"). You studied triangles, circles, lines, and so forth, and learned all sorts of formulas for area and volume and angle.

When a mathematician thinks about a triangle, she is not thinking about any particular triangle in real life. She is thinking about a "pure" triangle, one that does not exist in the real world, but that exists in the context of the axioms of geometry and the rules of logic. She is not interested in the properties of a real triangle, but in the properties of this idealized, abstract triangle. Of course, she can draw a triangle on a piece of paper, but that is just a representation of the thing she is actually thinking about. After all, the triangle on the paper will not have perfectly straight sides, and the angles might not add up to 180 degrees. This drawing is only a model or avatar for the abstract mathematical triangle, which does have three perfectly straight sides and whose angles do add up to exactly 180 degrees.

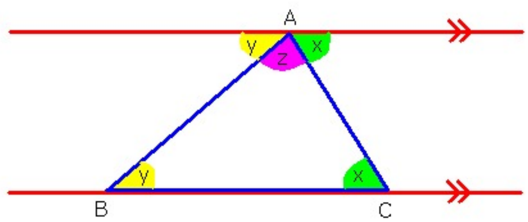

FIGURE 1. In a triangle, the interior angles always add up to 180 degrees

For another example, let's turn to the branch of mathematics called number theory (this would fall under what the Wikipedia article calls "quantity"). Number theory is concerned with the positive integers (1,2,3,4,...). Number theorists look for patterns in the integers, formulate interesting questions about them, and try to determine the deeper structure hidden in this seemingly-simple list of numbers. It is very important to realize that these numbers are studied as objects in and of themselves, with no reference to counting things in real life.

When I think about the number five, for instance, I am not thinking about five apples, five pencils, or five of anything else. Instead, I'm thinking about the abstract object "five" or "pure five-ness," if you will (this is not nearly as Platonic as it sounds). I can think about the properties of the number five and how it interacts with other numbers without thinking about counting anything at all.

The beginning of the second paragraph of the Wikipedia quotation is crucial to understanding what math is and what a mathematician does:

Mathematicians seek out patterns and use them to formulate new conjectures. Mathematicians resolve the truth or falsity of conjectures by mathematical proof.

For an illustration of this, let's go back to Euclidean geometry and the angles of a triangle, as in Figure 1. How do we know that the angles of a triangle add up exactly to 180 degrees? Well, we can prove it (though though won't here). We start with the axioms, or statements that we take as given truths, of Euclidean geometry (of which there are five). One of them is that we can always draw a straight line between any two points. Another is that we can draw circles with any given center and a radius as large or as small as we like.

Beginning with these axioms, and proceeding with chains of logical deductions, one eventually arrives at the theorem that the angles of a triangle (any triangle!) add up to 180 degrees. A *theorem* is any fact that has been deduced via rigorous mathematical arguments from the given axioms.

The notion of proof is central to mathematicians. Nothing can be taken for granted. Before we accept something as true, it must be proven to be true by correct logical arguments. Empirical observations are good enough for scientists, but not for mathematicians.

As an example of this principle, let us recall the Pythagorean theorem from Euclidean geometry (see Figure 2). It says that, if we have a right triangle with legs of length a and b and hypotenuse of length c, then

$$a^2 + b^2 = c^2.$$

Of course, we all believe this is true because we remember learning about it in our high school geometry classes, but what if you had never heard of this Pythagoras fellow before? What if someone gave you a bunch of right triangles and their side lengths, and you just started experimenting? Say you have right triangles with side lengths (3,4,5) and (5,12,13) and (8,15,17) and so on. On every triangle you check the square of the hypotenuse and the sum of the squares of the other two sides, and, sure enough, they come up equal every time. After you had checked four or five triangles you would probably be convinced that this property must hold for every right triangle.

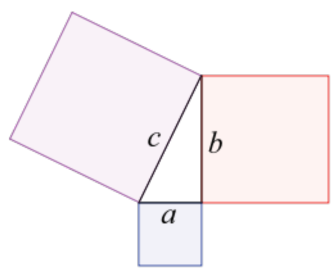

FIGURE 2. Visual representation of the Pythagorean theorem

Is this a proof? No, of course not. Perhaps there is some bizarre right triangle out there where this property fails, and you just gave up before getting to it? You quickly see that even if you checked triangles day in and day out for the rest of your natural life, it wouldn't be enough, because you could never rule out that there isn't a wacky triangle that you haven't checked yet. Of course, after checking so many triangles you would be quite convinced of the truth of the Pythagorean theorem, but you would not necessarily be any closer to seeing why it is true. What is required is a mathematical proof that deals with all right triangles at once.

WHAT IS RESEARCH?

Let's just give a definition.

Research: the act of trying to find the answers to interesting questions.

Notice that this definition applies to all sorts of topics, not just mathematics. Let's agree on this definition (even if you don't), because it helps us get where we really want to go, which is understanding what math research is. Given this definition, we would define math research to be the act of trying to find answers to interesting math questions.

Now that's all well and good, but, as you learned in your dreadful math classes in school, no one ever understood something by staring at an unmotivated definition. To give a bit of motivation and illustration, we embark on an extended meditation.

Let's go back to numbers, which we mentioned above. On their own, numbers aren't very interesting. After all, what does the number seven do, exactly? It doesn't seem to do anything at all; it just *is*.

Very well. The triangles we talked about above didn't do anything either: they just "triangled." We had to be creative and think of a question to ask, like if there is some relationship between the square of the hypotenuse and the other two sides. Once we had a question in hand, we could start doing things.

Similarly, the number seven just *is*, but we can ask just what sort of *is* it does. What makes seven different from six, say? Well, there are many things that make them different, but let's focus on one specific property. We see that six can be written

$$6 = 2 \times 3.$$

In other words, we can write six as the product of two numbers, both of which are strictly less than six. On the other hand, we quickly see that the only ways to write seven are

$$7 = 1 \times 7 = 7 \times 1.$$

We call a positive integer *prime* if it can only be written as the product of one and itself (it is a matter of convention that one is not a prime number). Thus the numbers 7, 31, and 257 are prime, but 6, 25, and 1001 are not.

This is very interesting stuff. Now that we have the notion of a prime number, we can ask all sorts of questions. Are there infinitely many prime numbers? Are they all odd except for the prime number 2? How many prime numbers are there up to 100? 1000? a billion?

Notice what we did here. We started with a collection of objects, in this case the positive integers. We didn't really know what to do, so we just started messing around with the numbers, and we isolated a property of "prime-ness" that applies to some numbers but not others. Once we have a property or definition like this in mind, a little bit of creativity yields many interesting questions and avenues of inquiry.

With our definition of prime numbers in hand, let's dig a bit more deeply and try to understand mathematical research through the lens of one very important mathematical theorem, the *prime number theorem*.

Brief Detour: Logarithms

In order to really understand what the prime number theorem (henceforth PNT) is about, we need to recall the notion of *logarithms*. You might have terrible memories of the logarithms that tormented you in your earlier years, but there is really nothing to fear. The common logarithm, or logarithm to base 10, we write as $\log_{10}$. This common logarithm describes a rule: you give me a number n, like 4 or 183 or 1,203,839,211, and $\log_{10}$ basically tells you how many decimal digits n has (there is more to it than this, of course, but this is an easy enough lie to swallow).

Now, mathematicians don't usually work with the logarithm to base 10. They work with something that is seemingly more complicated, but which is actually preferable for a multitude of reasons. We call this the natural logarithm, or just log. Basically, we can calculate $\log n$, for some number n, by first calculating $\log_{10} n$ and then multiplying by about

2.3 (I am indebted to Timothy Gowers' book *Mathematics: A Very Short Introduction* for this viewpoint on logarithms).

Let's do a quick example. Suppose I have the number $n = 5,908,081,112$. In order to calculate $\log n$, the first thing I should do is calculate $\log_{10} n$, that is, count how many digits n has. We easily see that n has 10 digits, so $\log_{10} n \approx 10$. We then multiply by 2.3, so $\log(5,909,081,112) \approx 23$.

When you were learning about logarithms in school they might have seemed more complicated than this, but there's really nothing to it. (Part of the reason students do so dreadfully in mathematics classes is that no one ever bothers to give some reasonable intuition about the concepts being studied.)

PNT: A History

With a knowledge of logarithms in hand, we are ready to understand the PNT, one of the most beautiful theorems in mathematics.

The story of the PNT begins, as many good mathematical stories do, with a young Carl Friedrich Gauss. He was born in Brunswick, Germany, in 1777, and quickly became known as a prodigy. He is arguably the most important and influential mathematician that ever lived.

When Gauss was 15 or 16 years old, he spent many hours making tables of prime numbers and investigating how the prime numbers are distributed (this was before computers, remember). On the basis of these numerical computations, he made the rather remarkable conjecture that an integer near n has roughly a $\frac{1}{\log n}$ chance of being prime.

Prime number theorem
(illustrated by selected values n from 10^2 to 10^{14})

n	$\pi(n)$ = number of primes less than or equal to n	$\frac{\pi(n)}{n}$ = proportion of primes among the first n numbers	$\frac{1}{\log n}$ = predicted proportion of primes among the first n numbers
10^2	25	0.2500	0.2172
10^4	1,229	0.1229	0.1086
10^6	78,498	0.0785	0.0724
10^8	5,761,455	0.0570	0.0543
10^{10}	455,052,511	0.0455	0.0434
10^{12}	37,607,912,018	0.0377	0.0362
10^{14}	3,204,941,750,802	0.0320	0.0310

FIGURE 3. Some data on prime numbers

We get a better feel for this $\frac{1}{\log n}$ business by looking at Figure 3. The primes start out fairly dense: fully 25 percent of the numbers less than 100 are prime. As we go higher, the primes thin out. We see from Figure 3 that about 12 percent of numbers less than 10,000 are prime, and by the time we get up to a million a little less than eight percent of numbers are prime. Comparing the third and fourth columns, that is, the actual versus expected proportions, we see that $\frac{1}{\log n}$ provides a reasonably good guess. It seems reasonable to suppose that if we look at larger and larger numbers, then $\frac{1}{\log n}$ will become a better and better guess for the proportion of primes up to n.

You might notice the funny-looking π symbol in Figure 3. This has nothing to do with the mathematical constant π. Rather, it has become traditional to use $\pi(n)$ to denote the prime counting function, i.e. $\pi(n)$ counts the number of integers less than n that are prime numbers. We can see the values of $\pi(n)$ for various values of n in the first column of the table.

Another way to write Gauss' conjecture is that, if n is very large, then

$$\pi(n) \approx \frac{n}{\log n}.$$

This statement, in a somewhat more exact form than we have given here, is now a known fact, and is what we mean when we talk about the PNT.

Gauss never published his thoughts about the prime-counting function $\pi(n)$. It was not until 1808 that the French mathematician Legendre put into print a conjecture about $\pi(n)$. On the basis of numerical calculations, he suggested that

$$\pi(n) \approx \frac{n}{\log n + 1.08366}.$$

We can see that Legendre's guess does not match Gauss' guess, and of course by the PNT we now know that Gauss was right and Legendre was wrong.

Dirichlet, in 1837, and Chebyshev, in the early 1850s, made significant advances on the PNT and related questions. However, the real turning point in the unfolding saga of the PNT came in 1860, when Bernhard Riemann, a student of Gauss, published his 8-page memoir "On the number of primes less than a given magnitude." Riemann's paper was epochal, a paradigm shift which introduced tools and vistas of unmatched power. In this paper, Riemann laid out a program for attacking the PNT. He supplied proofs of some facts, but for other things he could only give sketches of proof or guess at their truth. Unfortunately, it is impossible to describe in a note such as this his contributions and insights. Suffice it to say, Riemann's ideas hinged on analyzing his now-eponymous "zeta function."

In 1896, two French mathematicians Jacques Hadamard and Charles Jean de la Vallée-Poussin, succeeded in proving the PNT independently. Their work built on the fundamental ideas of Riemann, and introduced further new ideas that continue to be useful to this day.

Even though the PNT was proved more than a century ago, there are still many difficult and fascinating questions about prime numbers that have yet to be solved. The achievements of these early pioneers continue to serve as the foundation on which present researchers build.

Conclusion

There are several things we can learn about math research from our discussion of the history of the PNT. The most obvious and striking lesson is that it required the combined efforts of many intelligent people over the period of many years in order to arrive at a solution. This stands in sharp contrast to the popular stereotype of mathematicians as geniuses working in isolation. Indeed, we see that mathematics research is rather a collective endeavor, each generation of researchers expanding upon the work of those that have gone before.

Another lesson we learn is that research is hard! Gauss' conjecture was very simple to state and understand, but its solution was anything but simple. It required a century of research and scores of brilliant ideas from different mathematicians in order to give a proof of Gauss' conjecture. Thus, we see that simple questions (and the most interesting questions are always simple) very often tend to have difficult answers. Contemporary mathematics is flooded with such simple questions, the vast majority of which are still open. An important skill any professional mathematician must develop is the skill of asking the right questions, that is, questions that are hard enough to be interesting, but not so hard that there is still a hope of finding an answer.

Well, there you have it. Now you know a little bit more about math research. We learned that math research is the process of trying to find answers to interesting math questions. These questions might have applications to daily life, or they might be studied solely due to their intrinsic interest. We learned that mathematicians are in the business of proving theorems, and that empirical evidence, while helpful for formulating conjectures, is not a satisfactory substitute for rigorous proof. Above all, we learned that math research is usually very challenging, but that it can also be immensely rewarding, intellectually satisfying, and intensely beautiful. □

Picture Credits.

I did not create any of the three pictures used in this note. The picture in Figure 1 was created, I believe, by Sanket Alekar, and can be found at

https://www.quora.com/Why-do-the-angles-in-a-triangle-add-up-to-180-and-not-200-or-some-other-number

The picture in Figure 2 came from the Wikipedia page on the Pythagorean theorem (accessed June 5, 2017), and can be found at

https://en.wikipedia.org/wiki/Pythagorean_theorem

The picture in Figure 3 came from the Encyclopedia Britannica page on the prime number theorem, and can be found at

https://www.britannica.com/topic/number-theory/Prime-number-theorem.

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