Kuratowski’s Theorem

A Kuratowski graph is a subdivision of $K_5$ or $K_{3,3}$. It follows from Euler’s Formula that neither $K_5$ nor $K_{3,3}$ is planar. Thus every Kuratowski graph is nonplanar. Our goal is to prove the following classic theorem.

**Theorem 1** (Kuratowski, 1930). A graph $G$ is planar if and only if $G$ does not contain a Kuratowski subgraph.

The “only if” part is already proved. Let us prove the “if” part.

**Claim 2.** For every graph $G$ and any $xy \in E(G)$, if $G$ does not contain a Kuratowski subgraph, then $G/xy$ also doesn’t.

**Proof.** Suppose that $G/xy$ contains a Kuratowski subgraph $H$. Let $z$ be the vertex resulting from contracting $x$ with $y$. If $z \notin V(H)$, then $H$ is a Kuratowski subgraph of $G$. If $z \in V(H)$ but is not a branch vertex of $H$, then we can obtain a Kuratowski subgraph $H'$ of $G$ by replacing $z$ in $H$ with either $x$, or $y$, or $\{x, y\}$. The same holds if $z$ is a branch vertex of $H$, and at most one edge of $H$ incident with $z$ is incident with $x$ in $G$. Thus the remaining case is that $H$ is a subdivision of $K_5$ and exactly two edges of $H$ incident with $z$ are incident with $x$ in $G$ (see Fig. 1 (left)).

![Figure 1](image)

Then $G$ contains a subdivision of $K_{3,3}$ as in Fig. 1 (right). □

First, we will prove a stronger statement for 3-connected graphs. A convex embedding of a planar graph $G$ is one in which every edge of $G$ forms a straight segment and every face (including the outer face) is a convex polygon. Not every planar graph has a convex embedding; for example, $K_{2,4}$ has not.

**Theorem 3** (Tutte). Every 3-connected graph with no Kuratowski subgraph has a convex embedding in the plane with no three vertices on a line.

**Proof.** By induction on $n := |V(G)|$. If $n \leq 4$, then the only 3-connected graph is $K_4$, and $K_4$ has such embedding.

Suppose the theorem holds for all graphs with at most $n - 1$ vertices. Let $G$ be any $n$-vertex 3-connected graph with no Kuratowski subgraph. By Contraction Lemma (7.2.7 in the book), $G$ has an edge $xy$ such that $H := G/xy$ is 3-connected. By Claim 2, $H$ has no
Kuratowski subgraph. So by the IH, $H$ has a convex embedding in the plane with no three vertices on a line. Fix such an embedding. Let $z$ be the result of contracting $xy$ and $H'$ be obtained from $H$ by deleting all edges incident with $z$. Since $H' - z$ is 2-connected, the face $C$ of $H'$ containing $z$ is a cycle. Let $x_1, \ldots, x_k$ be the neighbors of $x$ on $C$ in cyclic order. If there is some $i$ such that all neighbors of $y$ on $C$ are in the portion of $C$ between $x_i$ and $x_{i+1}$, then we can obtain a convex embedding of $G$ with no three vertices on a line by placing $x$ into the position of $z$ and placing $y$ very close to $x$. If this does not happen, then either (a) $y$ and $x$ have 3 common neighbors, say $u, v, w$, or (b) for some $i < j$, $y$ has a neighbor $v$ on $C$ between $x_i$ and $x_j$ (in clockwise order) and a neighbor $u$ between $x_j$ and $x_i$.

In Case (a) we have a $K_5$-subdivision and in Case (b) we have a $K_{3,3}$-subdivision. □

In order to prove Theorem 1, it is now enough to show the following.

**Lemma 4.** If $G$ has the fewest vertices among the nonplanar graphs with no Kuratowski subgraphs, then $G$ is 3-connected.

**Proof.** We need the following simple observation:

(**) If $F$ is a face in an embedding of a graph $G$ in the plane, then there is an embedding of $G$ in the plane where $F$ the outer face.

If $G$ is disconnected, then by the minimality of $G$, each of its components could be embedded in the plane. The union of these embeddings will be an embedding of $G$. Suppose $G$ has a cut vertex $x$ and $H$ is a component of $G - x$. Let $H_1 = G[V(H) + x]$ and $H_2 = G - H$. By the minimality of $G$, each of $H_1$ and $H_2$ could be embedded in the plane. Then by (**), each of $H_1$ and $H_2$ has an embedding in the plane such that $x$ is on the outer face. Stretching each of these embeddings so that each of the graphs is in one half-plane passing through $x$, we can then glue them into an embedding of $G$.

Suppose now that $G$ is 2-connected and that sets $V_1, V_2 \subseteq V(G)$ and vertices $x, y$ are such that $V_1 \cup V_2 = V(G)$, $V_1 \cap V_2 = \{x, y\}$ and there are no edges between $V_1 - x - y$ and $V_2 - x - y$. For $i = 1, 2$, let $G_i$ be the graph obtained from $G[V_i]$ by adding edge $xy$. If both $G_1$ and $G_2$ are planar, then by (**), there are their embeddings with edge $xy$ on the outer face. Again, we can stretch these embeddings so that we can glue them along $xy$ and get an embedding of $G$. Thus we may assume that $G_1$ is not planar. By the minimality of $G$, $G_1$ contains a Kuratowski subgraph $H$. Since $G$ does not contain Kuratowski subgraphs, $H$ contains edge $xy$. So we can get a Kuratowski subgraph $H'$ of $G$ from $H$ be replacing $xy$ with an $x, y$-path in $G[V_2]$. Such an $x, y$-path exists, since $G$ is 2-connected and so each of $x$ and $y$ has a neighbor in every component of $G - x - y$. □