MATH 582, SPRING 2019 – PROBLEM SET 4

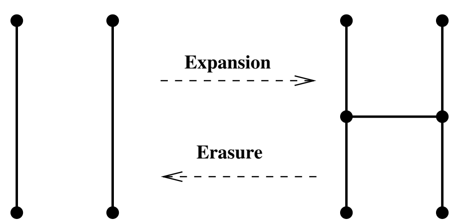
Do five of the six problems below. Due Wednesday, April 03.

1. A graph G is H-linked if for every injection $f: V(H) \to V(G)$, there is an H-subdivision in G that puts the vertex representing $v \in V(H)$ at $f(v)$ in G for all $v \in V(H)$. (a) Prove that if G is H-linked and H' is obtained from H by identifying a vertex x of degree 1 with another vertex, then G is H'-linked.

(b) Prove that if G is k-linked and H has k edges and no isolated vertices, then G is H-linked.

2. Let H be the graph consisting of two disjoint K_2 and three isolated vertices. What is the minimum connectivity that may have an H-linked graph? What is the maximum connectivity that may have a non-H-linked graph? (Hint: You may use all results stated (even without proofs) in Section 7.1 of the book.)

3. An *expansion* subdivides two edges and adds an edge joining the two new vertices. An *erasure* deletes an edge connecting two vertices of degree 3 and replaces each of the two obtained paths of length 2 by a single edge. (See the figure below.) Erasure is not allowed if it would produce multiple edges.



(a) Use 2-switches (recall Definition 6.2.6) and Theorem 6.2.7 to prove that every 3-regular graph can be obtained from K_4 by a sequence of expansions and erasures.

(b) For every n divisible by 8, construct a 3-regular 2-connected n-vertex graph that cannot be obtained from a smaller 3-regular graph by expansion.

4. Prove that applying the expansion operation of Problem 3 to a 3-connected graph yields a 3-connected graph. Obtain the Petersen graph from K_4 by expansions. (Comment: Tutte proved that a 3-regular graph is 3-connected if and only if it arises from K_4 by a sequence of expansions.)

5. Given a graph G, a *generalized vertex k-split* forms a graph H by replacing one vertex x with a clique $X = \{x_1, \dots, x_r\}$ such that $\bigcup_{i=1}^{r} N_H(x_i) = N_G(x) \cup \{x_1, \dots, x_r\}$ and $d_H(x_i) \geq k$ for all $1 \leq i \leq r$.

(a) Prove that if G is k-connected and

$(*)$ for every $T \subseteq X$, the number of neighbors of T in $N_G(x)$ is at least $k - r + |T|$,

then H is k-connected.

(b) Conclude that replacing a vertex x in a 3-connected 3-regular graph with a triangle and a matching joining the new vertices with the neighbors of x yields a 3-connected 3-regular graph.

6. *Applications of Mader's Theorem.*

(a) Prove that a minimally k-connected graph has at least k vertices of degree k. This improves the bound of Bollobás when $n(G) < 2k$.

(b) Let S be a separating k-set in a minimally k-connected graph G, and let C be a component of $G - S$. Prove that C contains a vertex of degree k in G. (Hint: consider the edge between a leaf of C and its neighbor.)