MATH 582, SPRING 2019 – PROBLEM SET 3

Do five of the six problems below. Due Wednesday, March 13.

1. (Bollobás' result) (a) Prove that each graph G with $\delta(G) \geq k-1$ contains each tree with k vertices.

(b) Let $n > k$ and $m > (k-1)(n-k/2)$. Prove that every n-vertex graph with at least m edges contains a subgraph with minimum degree at least k.

(c) Let $s := \lfloor n/\sqrt{2} \rfloor$. For $k = 2, 3, \ldots, s$, let T_k be a tree with k vertices. Using (b) and (a) show that $T_2, T_3, \ldots, T_s$ pack into K_n. (Hint: Pack larger trees first.)

2. Find necessary and sufficient conditions for a graphic sequence to be the degree sequence of a connected (simple) graph.

3. Prove that the degree sequence of a graph with at least four edges is edge-reconstructible. (Hint: Start from the number of vertices of maximum degree.) Use this to prove that
(a) every graph with at least four edges and degrees of all vertices of the same parity is edge-reconstructible;
(b) the Edge-Reconstruction Conjecture holds for graphs with exactly four edges.

4. Let G be an n-vertex graph with a non-trivial automorphism group and $e(G) \geq \log_2(n!) \geq 4$. Prove that G is edge-reconstructible.

5. *Reconstruction of non-strong tournaments.* Prove the following statements about tournaments having *at least five vertices* to prove that non-strong tournaments, like disconnected graphs, are reconstructible. (*Strong* means strongly connected.)

a) In a strong tournament, each vertex appears in a cycle of each length.

b) A tournament is strong if and only if at least 2 vertex-deleted subtournaments are strong. (Hint: use part (a) for a short proof.)

c) Tournaments with sinks (vertices of outdegree 0) are reconstructible.

d) Non-strong tournaments without sinks are reconstructible.

6. Prove that
(a) Every n-vertex graph ($n \geq 3$) such that at least $n-2$ cards in the deck are disconnected is reconstructible;
(b) Every n-vertex graph ($n \geq 5$) with maximum degree $n-1$ is edge-reconstructible.