MATH 582, SPRING 2019 – PROBLEM SET 1

Do five of the six problems below. Due Wednesday, February 6.

1. Using the Directed Matrix Tree Theorem, find the number of spanning in-trees and out-trees with the root 6 in the picture below on the left.

2. Using the Matrix Arborescence Theorem, find in the same (left) picture the number of spanning out-trees with the root 1 in which 1 has out-degree 2. Find in the right picture the number of spanning out-trees with the root 2 in which 5 has out-degree 2.

3. Using the BEST Theorem, find the number of Eulerian circuits in the graph in the picture above on the right.

4. Let \( f(r, s) \) be the number of trees on \( [r + s] \) having partite sets of sizes \( r \) and \( s \). Using Prüfer codes, prove that if \( r \neq s \), then \( f(r, s) = \binom{r+s}{s} s^{-1} r^{-1} \). What is the formula when \( r = s \)? (Hint: Show first that the Prüfer code for such a tree has \( r-1 \) terms from one part and \( s-1 \) terms from the other.)

5. Let \( F = (A, B; E) \) be a bipartite graph with parts \( A \) and \( B \). For \( r \geq 1 \), let \( B_r \) be the subset of vertices in \( B \) of degree at least \( r \). Prove that there is a subgraph \( F' \) of \( F \) for which both of the two properties below hold:
   (a) \( d_{F'}(v) \leq \left\lceil \frac{d_F(v)}{r} \right\rceil \) for all \( v \in A \);
   (b) \( d_{F'}(w) = 1 \) for all \( w \in B_r \) and \( d_{F'}(w) = 0 \) for all \( w \in B - B_r \).
   (Hint: Use Hall’s Theorem.)

6. **Binet-Cauchy Formula.** Let \( C = AB \), where \( A \) is an \( n \times m \) matrix and \( B \) is an \( m \times n \) matrix. Given \( S \subseteq [m] \) with \( |S| = n \), let \( A_S \) denote the \( n \times n \) submatrix of \( A \) whose columns are indexed by \( S \) and let \( B_S \) denote the \( n \times n \) submatrix of \( B \) whose rows are indexed by \( S \). Prove that \( \det C = \sum_S \det A_S \det B_S \), where the summation extends over all \( n \)-element subsets of \([m]\). (This completes the proof of the Matrix Tree Theorem.)
   (Hint: Consider the matrix equation \( \begin{pmatrix} I_m & 0 \\ A & I_n \end{pmatrix} \begin{pmatrix} -I_m & B \\ A & 0 \end{pmatrix} = \begin{pmatrix} -I_m & B \\ 0 & AB \end{pmatrix} \).)