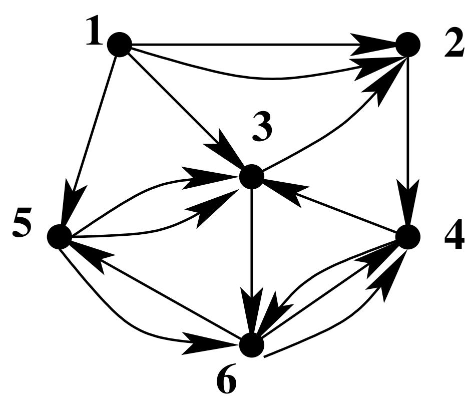
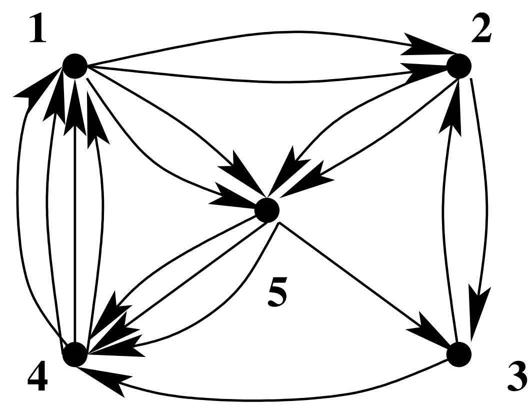
MATH 582, SPRING 2019 – PROBLEM SET 1

Do five of the six problems below. Due Wednesday, February 6.

1. Using the Directed Matrix Tree Theorem, find the number of spanning in-trees and out-trees with the root **6** in the picture below on the left.

2. Using the Matrix Arborescence Theorem, find in the same (left) picture the number of spanning out-trees with the root **1** in which **1** has out-degree 2. Find in the right picture the number of spanning out-trees with the root **2** in which **5** has out-degree 2.





3. Using the BEST Theorem, find the number of Eulerian circuits in the graph in the picture above on the right.

4. Let $f(r,s)$ be the number of trees on $[r+s]$ having partite sets of sizes r and s. Using Prüfer codes, prove that if $r \neq s$, then $f(r,s) = \binom{r+s}{s} s^{r-1} r^{s-1}$. What is the formula when $r = s$? (Hint: Show first that the Prüfer code for such a tree has $r-1$ terms from one part and $s-1$ terms from the other.)

5. Let $F = (A, B; E)$ be a bipartite graph with parts A and B. For $r \geq 1$, let B_r be the subset of vertices in B of degree at least r. Prove that there is a subgraph F' of F for which both of the two properties below hold:
(a) $d_{F'}(v) \leq \left\lceil \frac{d_F(v)}{r} \right\rceil$ for all $v \in A$;
(b) $d_{F'}(w) = 1$ for all $w \in B_r$ and $d_{F'}(w) = 0$ for all $w \in B - B_r$.
(Hint: Use Hall's Theorem.)

6. **Binet-Cauchy Formula.** Let $C = AB$, where A is an $n \times m$ matrix and B is an $m \times n$ matrix. Given $S \subseteq [m]$ with $|S| = n$, let A_S denote the $n \times n$ submatrix of A whose columns are indexed by S and let B_S denote the $n \times n$ submatrix of B whose rows are indexed by S. Prove that $\det C = \sum_S \det A_S \det B_S$, where the summation extends over all n-element subsets of $[m]$. (This completes the proof of the Matrix Tree Theorem.)
(Hint: Consider the matrix equation $\begin{pmatrix} I_m & 0 \\ A & I_n \end{pmatrix} \begin{pmatrix} -I_m & B \\ A & 0 \end{pmatrix} = \begin{pmatrix} -I_m & B \\ 0 & AB \end{pmatrix}$.)