On graph packing

Math 582, January 30, 2019

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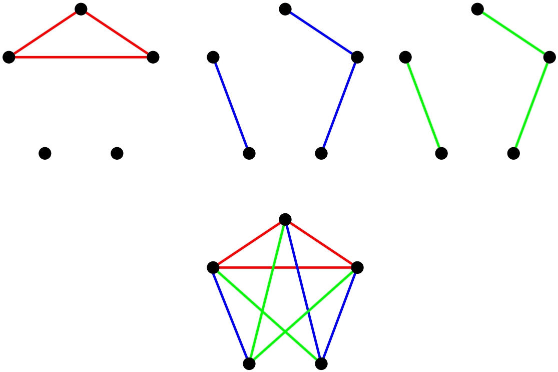
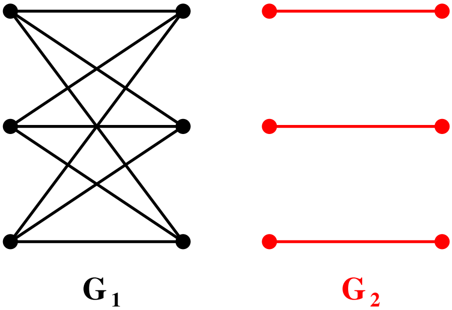
By definition, graphs G_1 and G_2 pack, if the complement, $\overline{G}_1$, of G_1 contains G_2

or, equivalently, the complement, $\overline{G}_2$, of G_2 contains G_1.

Example 1: Three graphs that pack



Example 2: Two graphs that do not pack



$\Delta(G_1)\Delta(G_2) = n/2.$

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- An n-vertex graph G has an equitable k-coloring $\Longleftrightarrow$ G packs with $H(n,k)$.

Some known theorems in the language of packing

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Hajnal-Szemerédi Theorem. Every n-vertex graph G with $\Delta(G) \leq k - 1$ packs with $H(n, k)$.

Extremal problems on graph packing

Bollobás and Eldridge, Sauer and Spencer, Catlin.

Results by Sauer and Spencer

Let G and H be n-vertex graphs.

Theorem 1. If $|E(G)| \leq n-2$ and $|E(H)| \leq n-2$, then G and H pack.

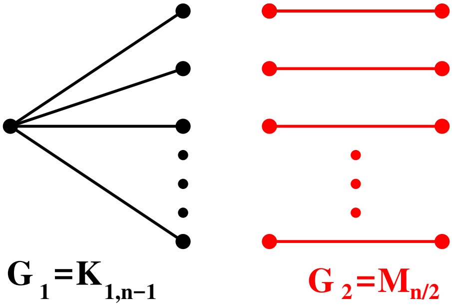
Theorem 2. If $|E(G)||E(H)| < \binom{n}{2}$, then G and H pack.

Corollary 1. If $|E(G)| + |E(H)| < \frac{3n}{2} - 1$, then G and H pack.

Theorem 3. If $\Delta(G)\Delta(H) < n/2$, then G and H pack.

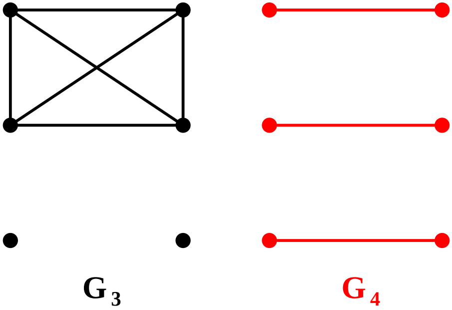
Conjecture 1. If $\Delta(G) = 2$ and $\Delta(H) < \frac{n}{3} - 1$, then G and H pack.

Example 3: Two graphs that do not pack



$|E(G_1)||E(G_2)| = \binom{n}{2}$ and $|E(G_1)| + |E(G_2)| = 3n/2 - 1$.
Also, $|E(G_1)| = n - 1$.

Example 4: Two graphs that do not pack

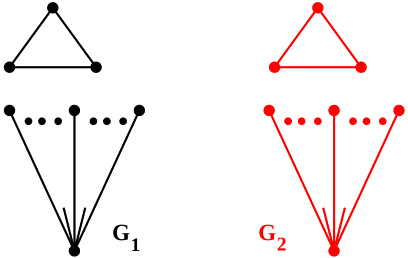


$$\Delta(G_3)\Delta(G_4) = n/2.$$

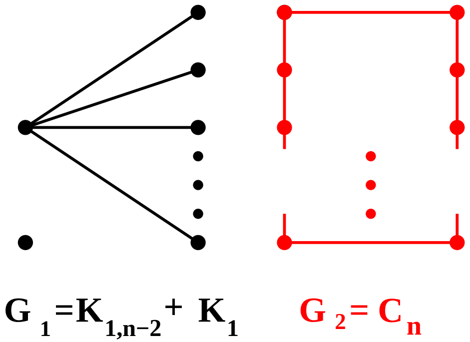
A result by Bollobás and Eldridge

Theorem 4. [Bollobás and Eldridge] Let $n > 10$. If G_1 and G_2 are n-vertex graphs such that $\Delta(G_1), \Delta(G_2) \leq n-2$ and $e(G_1) + e(G_2) \leq 2n-3$, then G_1 and G_2 pack.

Example 5:



Example 6: Two graphs that do not pack



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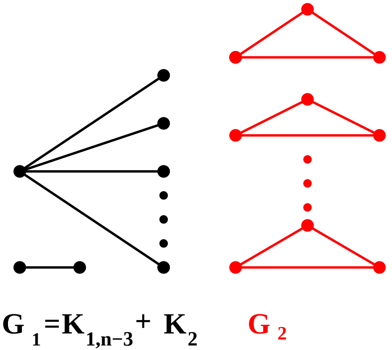

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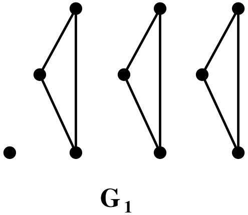
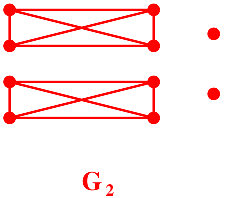
The Hajnal-Szemerédi Theorem is a partial case of it.





Theorem 5. [Kaul and A.K.] Let G_1 and G_2 be n-vertex graphs with $\Delta(G_1)\Delta(G_2) \leq n/2$. G_1 and G_2 do not pack if and only if one of them is a perfect matching and the other either is $K_{\frac{n}{2},\frac{n}{2}}$ with $\frac{n}{2}$ odd or contains $K_{\frac{n}{2}+1}$.

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Theorem 6. [Kaul, A.K., and Yu] Let G_1 and G_2 be n-vertex graphs with $\Delta(G_1), \Delta(G_2) \geq 300$. If

$$(\Delta(G_1)+1)(\Delta(G_2)+1) \leq 0.6n+1, \tag{1}$$

then G_1 and G_2 pack.

Gabor Kuhn.