Let $[k] := \{1, \ldots, k\}$. We consider simple graphs and follow the notation in [2]. An edge $k$-coloring of a graph $G$ is a mapping $\phi : E(G) \rightarrow [k]$ such that $\phi(e) \neq \phi(e')$ for any two edges $e, e'$ sharing a vertex. In other words, for every $1 \leq i \leq k$, the color class $\phi^{-1}(i)$ is a matching. The edge chromatic number, $\chi'(G)$, of $G$ is the smallest positive integer $k$ such that $G$ has an edge $k$-coloring. The classical result is:

**Theorem 1 (Vizing [1])** For every simple graph $G$, $\chi'(G) \leq \Delta(G) + 1$.

We present a proof of Theorem 1 due to Ehrenfeucht, Faber and Kierstead.

**Proof of Theorem 1:** For $\Delta(G) = D \in \{1, 2\}$ the proof is easy. Let $D \geq 3$. We use induction on the number of vertices of degree at least 2. Let $G$ be a minimum counter-example with $\Delta(G) = D \geq 3$. Let $v \in V(G)$ with $d_G(v) \geq 2$ and let $X = \{x_1, \ldots, x_t\}$ be the set of neighbors of $v$ of degree at least 2. If $t \leq 1$, then let $G'$ be obtained from $G$ by deleting $v$ and its neighbors of degree 1. By the minimality of $G$, $G'$ has an edge-$(D + 1)$-coloring $\phi$ which we greedily extend to an edge-$(D + 1)$-coloring of $G$.

So, suppose $t \geq 2$ and now let $G'$ be obtained from $G$ by first deleting $v$ and its neighbors of degree 1, and then adding new vertices of degree 1 (if needed) so that the degree of each of $v_1, \ldots, v_t$ in $G'$ is $D - 1$. Since $G'$ has fewer vertices of degree 1 than $G$, by the minimality of $G$, $G'$ has an edge-$(D + 1)$-coloring $\phi$.

For every $v \in V(G')$, let $O(v) = O_\phi(v)$ denote the set of colors in $[D + 1]$ not used to color the edges incident to $v$. Then $|O(x_i)| = 2$ for all $i = 1, \ldots, t$.

For every color $\alpha \in [D + 1]$ and an edge-$(D + 1)$-coloring $\phi$ of $G'$, let $h_\phi(\alpha)$ the number of $O(x_i)$ containing $\alpha$. Among all edge-$(D + 1)$-colorings of $G'$, choose a coloring $\psi$ with the minimum $\sum_{j=1}^{D+1} h_\psi^2(j)$.

Let $G'(\alpha, \beta)$ denote the subgraph of $G'$ formed by the edges of colors $\alpha$ and $\beta$ in $\psi$. Our main claim is

**Claim 1** For every $\alpha, \beta \in [D + 1]$, $|h(\alpha) - h(\beta)| \leq 2$.

Proof: Suppose $\alpha, \beta \in [D + 1]$ and $h(\alpha) - h(\beta) \geq 3$. Then $G'(\alpha, \beta)$ contains a component that is a path $P$ starting from a vertex $x_i \in X$ with the first edge of color $\alpha$ either its second end is not in $X$ or it is in $X$ and the last edge of $P$ also has color $\alpha$. In the first case, $h(\alpha)$ decreases by 1 and $h(\beta)$ increases by 1. In the second case, $h(\alpha)$ decreases by 2 and $h(\beta)$ increases by 2. In both cases, $\sum_{j=1}^{D+1} h_\psi^2(j)$ decreases, contradicting the choice of $\psi$. □

Claim 1 immediately implies:

**Claim 2** For every $\alpha \in [D + 1]$, $h(\alpha) \leq 3$, and if there is $\alpha \in [D + 1]$ with $h(\alpha) = 3$, then $h(\beta) \geq 1$ for every $\beta \in [D + 1]$.

A $[\beta, \gamma]$-path in $G$ is a path whose edges are alternately colored with $\beta$ and $\gamma$.

**Case 1:** $h(\alpha) \leq 2$ for every $\alpha \in [D + 1]$. Let $H$ be the bipartite graph whose partite sets are $X$ and $[D + 1]$ and $x_i \alpha \in E(H)$ iff $\alpha \in O_\psi(x_i)$. Then $d_H(x_i) = 2$ for every $i = 1, \ldots, t$ and $d_H(\alpha) \leq 2$ for every $\alpha \in [D + 1]$. By Hall’s Theorem, $H$ has a matching $M$ covering $X$. If $M = \{x_1 \alpha_1, \ldots, x_t \alpha_t\}$, then we can color edge $vw_i$ with $\alpha_i$ for every $i = 1, \ldots, t$.

**Case 2:** $h(\gamma_1) = 3$ for some $\gamma_1 \in [D + 1]$. Then by Claim 2, each of the $D + 1$ colors is present in some $O(x_j)$. We then find colors for the edges incident with $v$ one by one.

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Step 1: Since $D + 1 > t$, $h(\alpha_1) = 1$ for some color $\alpha_1$. We may assume that $O(x_1) = \{\alpha_1, \beta_1\}$. We will color $vx_1$ with $\alpha_1$ and let $X_1 = X - x_1$. Furthermore, if $h(\beta_1) \geq 2$, then we let $h_1(\gamma) = \{h(\gamma), \text{ if } \gamma \neq \beta_1; h(\beta_1) - 1, \text{ if } \gamma = \beta_1\}$. If $h(\beta_1) = 1$, then there are two vertices $x, x' \in X_1$ such that $\gamma_1 \in O(x) \cap O(x')$. For at least one of them the $[\gamma_1, \beta_1]$-path starting at it ends not at $x_1$. After recoloring this path, $h(\gamma_1)$ decreases by 1 or 2 and $h(\beta_1)$ increases by 1 or 2.

Step $i (i = 2, \ldots, t)$: If $h_{i-1}(\alpha) \leq 2$ for every $\alpha \in [D + 1] - \{\alpha_1, \ldots, \alpha_{i-1}\}$, then we use Hall’s Theorem as in Case 1. If $h(\gamma_i) = 3$ for some $\gamma_i \in [D + 1] - \{\gamma_1, \ldots, \gamma_{i-1}\}$, then each of the $D + 1 - (i - 1)$ colors in $[D + 1] - \{\gamma_1, \ldots, \gamma_{i-1}\}$ is present in some $O(x_j)$. Then we essentially repeat Step 1 with $i$ in place of 1. □

References
