1. Problem 2.3.10 in the book. Prove that an \( n \)-tuple \((d_1, \ldots, d_n)\) of nonnegative integers with even sum is graphic if and only if for all disjoint sets \( I \) and \( J \) of indices,
\[
\sum_{i \in I} d_i + \sum_{j \in J} (n - 1 - d_j) \geq |I||J|.
\] (1)

2. A multigraph \( H \) is almost \( k \)-edge-connected if for at most one nontrivial partition \( \{A, B\} \) of \( V(H) \), the number of edges connecting \( A \) with \( B \) is less than \( k \). Prove that for every odd \( d \geq 7 \) every almost \( \lceil d/3 \rceil \)-edge-connected \( d \)-regular multigraph has a 3-factor.

   (a) Let \( G \) be the union of graphs \( F \) and \( H \). Prove that \( \chi(G) \leq \chi(F) \cdot \chi(H) \).
   (b) Let \( D \) be an orientation of a graph \( G \) with \( \chi(G) > rs \). Let \( f : V(D) \to \mathbb{R} \). Using (a) and the G-H-R-V Theorem prove that \( D \) has a path \( u_0 u_1 \ldots u_r \) with \( f(u_0) \leq f(u_1) \leq \ldots \leq f(u_r) \) or a path \( v_0 v_1 \ldots v_s \) with \( f(v_0) > f(v_1) > \ldots > f(v_s) \).
   (c) Use part (b) to conclude the Erdős-Szekeres Theorem claiming that every sequence of \( rs + 1 \) distinct real numbers has an increasing subsequence of size \( r + 1 \) or a decreasing subsequence of size \( s + 1 \).

4. Problem 3.1.41 in the book. Let \( G \) be a connected graph.
   (a) Let \( v \in V(G) \). Among all spanning trees of \( G \), let \( T \) be one that maximizes \( \sum_{u \in V(G)} d_T(v, u) \). Prove that every edge in \( G \) joins vertices belonging to a path in \( T \) starting at \( v \).
   (b) Prove that if \( \chi(G) > k \geq 2 \), then \( G \) has a cycle congruent to 1 modulo \( k \).
   (Hint: Define a coloring using the tree \( T \) in (a).)
   (c) Prove that a graph having no odd cycle of length more than \( 2j - 1 \) is \( 2j \)-colorable.

5. Let \( G \) be a graph with \( n \) vertices, \( m \) edges and \( \Delta(G) \leq 3 \). Prove that if \( G \) does not contain \( K_4 \), then \( G \) has a bipartite subgraph with at least \( m - n/3 \) edges. Present an infinite series of examples where \( n/3 \) is sharp.

6. Problem 3.2.19 in the book. Prove that if \( G \) is a \( k \)-critical graph, then the graph \( G' \) generated from \( G \) by applying Mycielski’s construction is \((k+1)\)-critical.