Petersen’s proof of his theorem

**Theorem 1 (Petersen)** For every positive integer \( k \), every \( 2k \)-regular multigraph can be decomposed into \( k \) 2-regular spanning subgraphs.

First, Petersen rephrased it in a formally more general form:

**Theorem 2 (Petersen)** For every positive integer \( k \), every multigraph \( G \) with maximum degree at most \( 2k \) can be decomposed into \( k \) spanning subgraphs \( G_1, \ldots, G_k \) with maximum degree at most 2.

**Proof of Theorem 2.** We use induction on \( e(G) \).

**Base of induction.** If \( \Delta(G) \leq 2 \), the claim is trivial.

**Induction step.** Let \( G \) be a multigraph with the fewest edges for which the theorem is not true for some \( k \) such that \( \Delta(G) \leq 2k \). By the induction base, \( k \geq 2 \). By minimality, \( G \) is connected.

**Case 1:** \( k = 2 \). Let \( G_0 \) be any connected \( 4 \)-regular multigraph containing \( G \) (if \( G \) is not \( 4 \)-regular itself, we can obtain \( G_0 \) by taking two disjoint copies \( G', G'' \) of \( G \) and connecting each \( v \in V(G') \) with its copy in \( G'' \) by \( 4 - d(v) \) edges). Then \( G_0 \) is Eulerian and has an even number \( 2|V(G_0)| \) of edges. So we can number the edges \( e_1, \ldots, e_{2t} \) of \( G_0 \) along an Eulerian circuit \( C \) in \( G_0 \) and let \( E_1 = \{ e_1, e_3, \ldots, e_{2t-1} \} \) and \( E_2 = \{ e_2, e_4, \ldots, e_{2t} \} \). Since consecutive edges in \( C \) have numbers of different parities and the parities of the first and last edges of \( C \) are distinct, every vertex \( v \) of \( G_0 \) is incident with two odd-numbered edges and two even-numbered edges. Thus the subgraphs \( H_1 \) and \( H_2 \) of \( G_0 \) spanned by \( E_1 \) and \( E_2 \), respectively, are 2-regular, i.e. we have a decomposition of \( G_0 \) into two 2-regular spanning subgraphs. By deleting the vertices and edges not belonging to a fixed copy of \( G \), we get a decomposition of \( G \) into two subgraphs with maximum degree at most 2.

**Case 2:** \( k \geq 3 \). If every edge of \( G \) is a loop, then since \( G \) is connected, it has only one vertex, and the claim is trivial. Otherwise, let \( e \) be any non-loop edge of \( G \). Suppose it connects vertices \( x \) and \( y \). By the minimality of \( G \), the multigraph \( G_0 = G - e \) has a decomposition into multigraphs \( G_1, \ldots, G_k \) of maximum degree at most 2. Since \( d_{G_0}(x), d_{G_0}(y) \leq 2k - 1 \), there are \( 1 \leq i, j \leq k \) such that \( d_{G_i}(x) \leq 1 \) and \( d_{G_j}(y) \leq 1 \). If \( j = i \), then adding \( e \) to \( G_i \) yields the needed decomposition for \( G \). Suppose \( j \neq i \). Then the multigraph \( G'' \) with \( E(G'') = E(G_i) \cup E(G_j) \cup \{ e \} \) has maximum degree at most 4. So by Case 1, \( G'' \) has a decomposition into \( G'_i \) and \( G'_j \) of maximum degree at most 2 each. Together with remaining \( k - 2 \) subgraphs \( G_k \), they yield a required decomposition for \( G \). \( \square \)