Extremal problems on cliques

Part 6.1
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Such an extremal problem considered by Turán in 1941 asks: What is $\text{ex}(K_{r+1}, n)$?

A natural candidate for an extremal graph is so called Turán graph, a balanced complete $r$-partite graph $T_{n,r}$: it has $n$ vertices partitioned into $r$ parts $V_1, \ldots, V_r$ with

$$\left\lfloor \frac{n}{r} \right\rfloor \leq |V_1| \leq |V_2| \leq \ldots \leq |V_r| \leq \left\lceil \frac{n}{r} \right\rceil.$$
If \( r \geq n \), then \( T_{n,r} = K_n \) (each part has 1 or 0 vertices).

**Theorem 6.1 (Turán):**

\[
\text{ex}(K_r + 1, n) = |E(T_{n,r})|
\]
If $r \geq n$, then $T_{n,r} = K_n$ (each part has 1 or 0 vertices).

Theorem 6.1 (Turán): $\text{ex}(K_{r+1}, n) = |E(T_{n,r})|$. 
Claim: If $n \geq r + 1$, then

$$|E(T_{n,r})| - |E(T_{n-r,r})| = (r - 1)(n - r) + \binom{r}{2}.$$
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|E(T_{n,r})| - |E(T_{n-r,r})| = (r - 1)(n - r) + \binom{r}{2}.
\]

Proof of claim: We get a copy \( H \) of \( T_{n,r} \) from a copy \( G \) of \( T_{n-r,r} \) with parts \( V_1, \ldots, V_r \) by adding to each part \( V_i \) a new vertex \( v_i \) adjacent to all the \( n - r - |V_i| \) vertices in \( G - V_i \) and to all "new" vertices \( v_j \) for \( j \neq i \). The total number of edges between "new" vertices is \( \binom{r}{2} \). Then the total number of the added edges is

\[
\binom{r}{2} + \sum_{i=1}^{r} (n - r - |V_i|) = \binom{r}{2} + r(n - r) - \sum_{i=1}^{r} |V_i|
\]

\[
= \binom{r}{2} + (r - 1)(n - r), \quad \text{as claimed.}
\]
Proof of Turán’s Theorem

The inequality \( \text{ex}(K_{r+1}, n) \geq |E(T_{n,r})| \) holds by the definition of \( \text{ex}(K_{r+1}, n) \). We prove the inequality

\[
\text{ex}(K_{r+1}, n) \leq |E(T_{n,r})| \tag{1}
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by induction on \( n \).
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**Base of induction:** $n \leq r$. Since $K_n$ has the most edges among the simple $n$-vertex graphs, and in our case $T_{n,r} = K_n$, the theorem holds for $n \leq r$. 
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**Induction step.** Suppose that (1) holds for all $n' \leq n - 1$ and let $G$ be a simple $n$-vertex graph with no $K_{r+1}$ and $|E(G)| = \text{ex}(K_{r+1}, n)$. 
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The inequality $\text{ex}(K_{r+1}, n) \geq |E(T_{n,r})|$ holds by the definition of $\text{ex}(K_{r+1}, n)$. We prove the inequality

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**Induction step.** Suppose that (1) holds for all $n' \leq n - 1$ and let $G$ be a simple $n$-vertex graph with no $K_{r+1}$ and $|E(G)| = \text{ex}(K_{r+1}, n)$.

Let $Q$ be the vertex set of a largest complete subgraph in $G$. Since $G$ has no $K_{r+1}$, $|Q| \leq r$. Let $Q'$ be any set of $r - |Q|$ vertices in $G - Q$, and let $F = Q \cup Q'$. If $|Q| = r$, then $F = Q$. 
Since $Q$ is maximum, every $z \in V(G) - F$ is adjacent to at most $r - 1$ vertices in $F$. Thus

\[
\text{the number of edges between } F \text{ and } V(G) - F \text{ is at most } (r - 1)(n - r). \tag{2}
\]
Since $Q$ is maximum, every $z \in V(G) - F$ is adjacent to at most $r - 1$ vertices in $F$. Thus

the number of edges between $F$ and $V(G) - F$ is at most $(r - 1)(n - r)$. (2)

Let $G' = G - F$. By construction, $G'$ is a simple $(n - r)$-vertex graph with no $K_{r+1}$. By the induction assumption, $|E(G')| \leq \text{ex}(K_{r+1}, n - r) = |E(T_{n-r}, r)|$. 

This proves the induction step and thus the theorem.
Since \( Q \) is maximum, every \( z \in V(G) - F \) is adjacent to at most \( r - 1 \) vertices in \( F \). Thus

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Let \( G' = G - F \). By construction, \( G' \) is a simple \((n - r)\)-vertex graph with no \( K_{r+1} \). By the induction assumption,

\[
|E(G')| \leq \text{ex}(K_{r+1}, n - r) = |E(T_{n-r,r})|.
\]

Since \( |E(G[F])| \leq \binom{r}{2} \), by (2) and the claim,

\[
|E(G)| \leq |E(G')| + \binom{r}{2} + (r - 1)(n - r)
\]

\[
\leq |E(T_{n-r,r})| + \binom{r}{2} + (r - 1)(n - r) = |E(T_{n,r})|.
\]

This proves the induction step and thus the theorem.
So, the problem for $K_r$ is solved. But there are many other graphs. An important advance was made in 1946.

Let $s \ast k$ denote $k, k, \ldots, k$ ($s$ times written $k$).

**Theorem 6.2 (Erdős and Stone, 1946):** Fix $r, s \in \mathbb{N}$ and $\epsilon > 0$. If $n$ is sufficiently large, then every $n$-vertex graph with $|E(T_{n,r})| + \epsilon n^2$ edges contains the complete $(r + 1)$-partite graph $K_{(r+1)\ast s}$ with parts of size $s$. 
So, the problem for $K_r$ is solved. But there are many other graphs. An important advance was made in 1946.

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The importance of Theorem 6.2 is in the following corollary.

Theorem 6.3 (Erdős and Simonovits, 1966): Let $\mathcal{F} = \{F_1, \ldots, F_h\}$ and $r = \min\{\chi(F_1), \ldots, \chi(F_h)\} - 1$. Then

$$\lim_{n \to \infty} \frac{\text{ex}(\mathcal{F}, n)}{n^2} = \frac{1}{2} \left(1 - \frac{1}{r}\right).$$
Proof of Theorem 6.3 modulo Theorem 6.2.

Since $T_{n,r}$ does not contain any $F_i$,

$$\text{ex}(\mathcal{F}, n) \geq |E(T_{n,r})| = \frac{n^2}{2} \left(1 - \frac{1}{r}\right) + O(n).$$
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Since $T_{n,r}$ does not contain any $F_i$,

$$\text{ex}(\mathcal{F}, n) \geq |E(T_{n,r})| = \frac{n^2}{2} \left( 1 - \frac{1}{r} \right) + O(n).$$

For the upper bound, suppose $\chi(F_1) = r + 1$ and $|V(F_1)| = s$. Fix any $\epsilon > 0$.

By Theorem 6.2, for $n \geq n_0(r, s, \epsilon)$, each $n$-vertex $G$ with at least $|E(T_{n,r})| + \frac{\epsilon}{4} n^2$ edges contains $K_{(r+1)^*s}$ and hence $F_1$.

But $|E(T_{n,r})| + \frac{\epsilon}{4} n^2 < \frac{n^2}{2} \left( 1 - \frac{1}{r} + \epsilon \right)$.
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Since \( T_{n,r} \) does not contain any \( F_i \),

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For the upper bound, suppose \( \chi(F_1) = r + 1 \) and \( |V(F_1)| = s \). Fix any \( \epsilon > 0 \).

By Theorem 6.2, for \( n \geq n_0(r, s, \epsilon) \), each \( n \)-vertex \( G \) with at least \( |E(T_{n,r})| + \frac{\epsilon}{4} n^2 \) edges contains \( K_{(r+1)*s} \) and hence \( F_1 \).

But \( |E(T_{n,r})| + \frac{\epsilon}{4} n^2 < \frac{n^2}{2} \left( 1 - \frac{1}{r} + \epsilon \right) \). \( \square \)

Remark. There are examples showing that we may need \( s \leq c \log n \). We will discuss a proof by Nikiforov.
Lemma 6.4 (Nikiforov) Fix \( r \geq 2 \) and \( 0 < \alpha < 1/2 \). Let 
\( F = (X, Y; E) \) be a bigraph with \( |X| = m \) and \( |Y| = n \). Let 
\( s = \lfloor \alpha^r \ln n \rfloor \). If 

\[
\text{(a) } s \leq \frac{\alpha}{2} m + 1 \quad \text{and} \quad \text{(b) } |E| \geq \alpha m n,
\]

then \( F \) contains \( K_{s,t} \) with \( s \) vertices in \( X \) and \( t \geq t_0 = \left\lceil n^{1-\alpha^r-1} \right\rceil \) vertices in \( Y \).
Lemma 6.4 (Nikiforov) Fix $r \geq 2$ and $0 < \alpha < 1/2$. Let $F = (X, Y; E)$ be a bigraph with $|X| = m$ and $|Y| = n$. Let $s = \lfloor \alpha^r \ln n \rfloor$. If

\begin{align*}
(a) & \quad s \leq \frac{\alpha}{2} m + 1 \quad \text{and} \quad (b) \quad |E| \geq \alpha m n,
\end{align*}

then $F$ contains $K_{s, t}$ with $s$ vertices in $X$ and $t \geq t_0 = \lceil n^{1-\alpha^r-1} \rceil$ vertices in $Y$.

**Proof.** Choose maximum $t$ s.t. $F$ contains $K_{s, t}$ with $s$ vertices in $X$ and $t$ vertices in $Y$. Suppose $t < t_0$.

If $t = 0$ then using (3) (a), $|E| \leq (s - 1)n \leq \frac{\alpha}{2} mn$ contradicting (3) (b). Thus $t \geq 1$. 
By the maximality of \( t \), the number of \((s+1)\)-tuples \((x_1, \ldots, x_s, y)\) s.t. \(\{x_1, \ldots, x_s\} \subseteq N(y)\) is at most \(t\binom{m}{s}\). Hence using convexity of \(f(x) = \binom{x}{s}\) and then (3) \((b)\),

\[
t\binom{m}{s} \geq \sum_{y \in Y} \binom{d(y)}{s} \geq n\binom{|E|/n}{s} \geq n\binom{\alpha m}{s}.
\]
By the maximality of $t$, the number of $(s + 1)$-tuples $(x_1, \ldots, x_s, y)$ s.t. $\{x_1, \ldots, x_s\} \subseteq N(y)$ is at most $t \binom{m}{s}$.

Hence using convexity of $f(x) = \binom{x}{s}$ and then (3) (b),

$$t \binom{m}{s} \geq \sum_{y \in Y} \binom{d(y)}{s} \geq n \binom{|E|/n}{s} \geq n \binom{\alpha m}{s}.$$ 

Hence (using the bound on $s$ and the definition of $s$)

$$t \geq n \frac{\alpha m}{m^s} \geq n \frac{\alpha m - s + 1}{m^s} \geq n \frac{\alpha^2 m^s}{m^s} = n \left(\frac{\alpha}{2}\right)^s \geq n^{1 + \alpha r \ln \frac{\alpha}{2}}.$$
By the maximality of $t$, the number of $(s+1)$-tuples $(x_1, \ldots, x_s, y)$ s.t. \(\{x_1, \ldots, x_s\} \subseteq N(y)\) is at most $t\binom{m}{s}$.

Hence using convexity of $f(x) = \binom{x}{s}$ and then (3) (b),

$$t\binom{m}{s} \geq \sum_{y \in Y} \binom{d(y)}{s} \geq n\left(\frac{|E|}{n}\right) \geq n\left(\alpha m\right).$$

Hence (using the bound on $s$ and the definition of $s$)

$$t \geq n\frac{(\alpha m)^s}{\binom{m}{s}} \geq n\frac{(\alpha m - s + 1)^s}{m^s} \geq n\frac{(\frac{\alpha}{2} m)^s}{m^s} = n\left(\frac{\alpha}{2}\right)^s \geq n^{1 + \alpha r \ln \frac{\alpha}{2}}.$$

Since $\frac{\alpha}{2} < \frac{1}{4}$ and $f(x) = x \ln x$ decreases on $(0, \frac{1}{e})$,

$$t \geq n^{1 + \alpha r \ln \frac{\alpha}{2}} > n^{1 + 2\alpha r^{-1} \cdot \frac{1}{4} \ln \frac{1}{4}} > n^{1 - \alpha r^{-1}} \geq t_0. \qed$$
Theorem 6.5 (Nikiforov) Fix $r \geq 1$ and $0 < \alpha < 1/2$. If an $n$-vertex graph $G$ contains $\geq \alpha n^r$ copies of $K_r$, then $G$ contains the complete $r$-partite $K_{(r-1)s,t}$, where $s = \lceil \alpha^r \ln n \rceil$ and $t = \left\lceil n^{1-\alpha^{r-1}} \right\rceil$. 
Theorem 6.5 (Nikiforov) Fix $r \geq 1$ and $0 < \alpha < 1/2$. If an $n$-vertex graph $G$ contains $\geq \alpha n^r$ copies of $K_r$, then $G$ contains the complete $r$-partite $K_{(r-1)s,t}$, where $s = \lfloor \alpha^r \ln n \rfloor$ and $t = \lceil n^{1-\alpha r^{-1}} \rceil$.

Proof. By induction on $r$.

Base: $r \leq 2$. If $r = 1$, then there is nothing to prove. Let $r = 2$. Construct bigraph $F = (X, Y; E)$ where $X$ and $Y$ are disjoint copies of $V(G)$, and for each $v_i v_j \in E(G)$, $E$ contains $x_i y_j$ and $x_j y_i$.

By definition, $s < \frac{\alpha}{2} \ln n$ and $|E(G)| \geq \alpha n^2$. Then $|E| \geq 2\alpha n^2$, and by Lemma 6.4, $F$ contains a copy $Q$ of $K_{s,t}$.

Since $F$ has no edges of the kind $x_i y_i$,

$$Q \subset G.$$
Induction step. Let $M$ be a family of $\geq \alpha n^r$ copies of $K_r$ in $G$.

Step 1: *Finding a good set of $r$-cliques.* For a family $Q$ of $r$-cliques in $G$, let $\mathcal{K}(Q)$ denote the family of $(r - 1)$-subcliques of these cliques.
Induction step. Let $M$ be a family of $\geq \alpha n^r$ copies of $K_r$ in $G$.

**Step 1: Finding a good set of $r$-cliques.** For a family $Q$ of $r$-cliques in $G$, let $\mathcal{K}(Q)$ denote the family of $(r-1)$-subcliques of these cliques.

Let $L_0 = M$ and for $j = 0, 1, \ldots$, if $\mathcal{K}(L_j)$ has a clique $S$ contained in $< \alpha n$ of $r$-cliques in $L_j$, then we obtain $L_{j+1}$ by deleting from $L_j$ all the cliques containing $S$. Let $L$ be a resulting family (it is not unique).

Since the number of deletions is at most $\binom{n}{r-1}$, we have deleted at most $\alpha n \binom{n}{r-1} < \frac{\alpha}{2} n^r$ cliques. Hence $|L| > \frac{\alpha}{2} n^r$. 
Step 2: *Applying induction.* Let $M' = K(L)$. Then

$$|M'| \geq \frac{r}{n}|L| \geq \frac{r\alpha}{2} \cdot n^{r-1}.$$ \hspace{1cm} (4)

Let $\alpha' = \frac{r\alpha}{2}$. Since

$$|M'| \leq \binom{n}{r-1} < \frac{n^{r-1}}{2},$$

(4) yields $\alpha' < \frac{1}{2}$. 

One of the parts is even larger.
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$|M'| \leq \binom{n}{r-1} < \frac{n^{r-1}}{2},$ (4) yields $\alpha' < \frac{1}{2}$.

By IH, $G$ contains $H = K_{(r-1)*m}$, where

$m = \left\lfloor \left(\frac{r\alpha}{2}\right)^{r-1}\ln n \right\rfloor$.

(One of the parts is even larger.)
Step 3: Adding a huge part. We view $V(H)$ as the set of $m$ disjoint $(r - 1)$-cliques. Let $X$ be the set of these $m$ cliques.
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Define an $(X, V(G))$-bigraph $F$ by having $Qy \in E(F)$ iff $Q + y$ is an $r$-clique in $L$. 

By Step 1, $d_F(Q) \geq \alpha n$ for each $Q \in X$, and so $|E(F)| \geq \alpha mn$. 

By definition, $s \leq \alpha r \ln n = \alpha (r\frac{\alpha}{2})^{r-1}\ln n < \alpha (\frac{2}{3})^2(\lfloor(r\frac{\alpha}{2})^{r-1}\ln n\rfloor + 1) < \alpha^2 (m + 1) < \alpha^2 m + 1$. 

So by Lemma 6.4, $F$ contains $K_s, t$ with the $s$-part in $X$. 

This corresponds to $K_{r-1}^s, t$ in $G$. 
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By definition,

$$s \leq \alpha r \ln n = \alpha \left(\frac{r\alpha}{2}\right)^{r-1} \left(\frac{2}{r}\right)^{r-1} \ln n$$

$$< \alpha \left(\frac{2}{3}\right)^2 \left(\left\lfloor \left(\frac{r\alpha}{2}\right)^{r-1} \ln n \right\rfloor + 1 \right) < \frac{\alpha}{2}(m + 1) < \frac{\alpha}{2}m + 1.$$
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Define an $(X, V(G))$-bigraph $F$ by having $Qy \in E(F)$ iff $Q + y$ is an $r$-clique in $L$.

By Step 1, $d_F(Q) \geq \alpha n$ for each $Q \in X$, and so

$$|E(F)| \geq \alpha mn.$$ 

By definition,

$$s \leq \alpha' \ln n = \alpha \left( \frac{r\alpha}{2} \right)^{r-1} \left( \frac{2}{r} \right)^{r-1} \ln n$$

$$< \alpha \left( \frac{2}{3} \right)^2 \left( \left\lceil \left( \frac{r\alpha}{2} \right)^{r-1} \ln n \right\rceil + 1 \right) < \frac{\alpha}{2} (m + 1) < \frac{\alpha}{2} m + 1.$$

So by Lemma 6.4, $F$ contains $K_{s,t}$ with the $s$-part in $X$.

This corresponds to $K_{(r-1)*s,t}$ in $G$. 

\(\square\)
Lemma 6.6 (Moon-Moser, 1962) For $s \geq 2$, let $k_s = k_s(G)$ denote the number of $s$-cliques in $G$. Then

$$\frac{k_{s+1}}{k_s} \geq \frac{1}{s^2 - 1} \left( s^2 \frac{k_s}{k_{s-1}} - n \right). \quad (5)$$

**Proof:** Let $A_1, \ldots, A_{k_s}$ be the $s$-cliques in $G$ and let $\alpha_j$ be the number of $(s + 1)$-cliques containing $A_j$.

Similarly, let $B_1, \ldots, B_{k_{s-1}}$ be the $(s - 1)$-cliques in $G$ and let $\beta_j$ be the number of $s$-cliques containing $B_j$. 
Lemma 6.6 (Moon-Moser, 1962) For $s \geq 2$, let $k_s = k_s(G)$ denote the number of $s$-cliques in $G$. Then

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Similarly, let $B_1, \ldots, B_{k_{s-1}}$ be the $(s - 1)$-cliques in $G$ and let $\beta_j$ be the number of $s$-cliques containing $B_j$.

We count in two ways the number $z$ of pairs $(A, U)$ s.t. $|A| = |U| = s$, $A$ is an $s$-clique, $G[U]$ is NOT a clique, and $|A \cap U| = s - 1$. 
Since for each $B_j$ there are $\beta_j$ ways to get an $s$-clique $A$ and $n - s + 1 - \beta_j$ ways to get a $U$,

$$z = \sum_{j=1}^{k_{s-1}} \beta_j(n - s + 1 - \beta_j).$$

On the other hand, for each $s$-clique $A$ and each $v \in V(G) - A$ having a non-neighbor $x \in A$, we can get an $U$ in $s - 1$ ways by deleting from $A + v$ a vertex in $A - x$. So,

$$\sum_{i=1}^{k_s} (s - 1)(n - s - \alpha_i) \leq \sum_{j=1}^{k_{s-1}} \beta_j(n - s + 1 - \beta_j). \quad (6)$$
Since for each $B_j$ there are $\beta_j$ ways to get an $s$-clique $A$ and $n - s + 1 - \beta_j$ ways to get a $U$,

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$$\sum_{i=1}^{k_s} (s - 1)(n - s - \alpha_i) \leq \sum_{j=1}^{k_{s-1}} \beta_j(n - s + 1 - \beta_j). \quad (6)$$

Since $\sum_{i=1}^{k_s} \alpha_i = (s + 1)k_{s+1}$, the LHS of (6) is $(n - s)(s - 1)k_s - (s^2 - 1)k_{s+1}$.

The RHS is maximized when all $\beta_j$ are equal to \[ \frac{1}{k_{s-1}} \sum_{j=1}^{k_{s-1}} \beta_j = s \frac{k_s}{k_{s-1}}. \]
So

\[(n - s)(s - 1)k_s - (s^2 - 1)k_{s+1} \leq s k_s \left( n - s + 1 - s \frac{k_s}{k_{s-1}} \right) .\]

Divide both parts by \(k_s\) and rearrange:
So

\[(n - s)(s - 1)k_s - (s^2 - 1)k_{s+1} \leq s k_s \left( n - s + 1 - s \frac{k_s}{k_{s-1}} \right).\]

Divide both parts by \(k_s\) and rearrange:

\[(n - s)(s - 1) - s(n - s + 1) + s^2 \frac{k_s}{k_{s-1}} \leq (s^2 - 1) \frac{k_{s+1}}{k_s}.\]

This proves (5).

□
So

\[(n - s)(s - 1)k_s - (s^2 - 1)k_{s+1} \leq s k_s \left( n - s + 1 - s \frac{k_s}{k_{s-1}} \right).\]

Divide both parts by \(k_s\) and rearrange:

\[(n - s)(s - 1) - s(n - s + 1) + s^2 \frac{k_s}{k_{s-1}} \leq (s^2 - 1) \frac{k_{s+1}}{k_s} .\]

This proves (5).

Lemma 6.7. If \(G\) is an \(n\)-vertex graph with

\[|E(G)| \geq \frac{n^2}{2} \left( 1 - \frac{1}{r} + \epsilon \right), \]

then \(k_{r+1}(G) \geq \alpha \cdot n^{r+1},\)

where \(\alpha = \frac{\epsilon}{(r+1)r^{r-1}}\).
**Proof:** Multiply both parts of (5) by \( \frac{s+1}{s} \) and regroup:

\[
\frac{(s + 1)(k_{s+1})}{s \cdot k_s} - \frac{n}{s} \geq \frac{s \cdot k_s}{(s - 1)k_{s-1}} - \frac{n}{s - 1}.
\] (7)
Proof: Multiply both parts of (5) by \( \frac{s+1}{s} \) and regroup:

\[
\frac{(s+1)(k_{s+1})}{s \cdot k_s} - \frac{n}{s} \geq \frac{s \cdot k_s}{(s-1)k_{s-1}} - \frac{n}{s-1}.
\] (7)

Denoting \( a_j = j \cdot k_j \) in (7), we get for \( 2 \leq s \leq r \),

\[
\frac{a_{s+1}}{a_s} - \frac{n}{s} \geq \frac{a_s}{a_{s-1}} - \frac{n}{s-1} \geq \ldots \geq \frac{a_2}{a_1} - \frac{n}{1} = \frac{2|E(G)|}{n} - n.
\]

Since the last expression is at least \( n \left( \epsilon - \frac{1}{r} \right) \), we get

\[
\frac{a_{s+1}}{a_s} \geq n \left( \epsilon - \frac{1}{r} + \frac{1}{s} \right).
\]
Proof: Multiply both parts of (5) by $\frac{s+1}{s}$ and regroup:

$$\frac{(s + 1)(k_{s+1})}{s \cdot k_s} - \frac{n}{s} \geq \frac{s \cdot k_s}{(s - 1)k_{s-1}} - \frac{n}{s - 1}.$$  \hfill (7)

Denoting $a_j = j \cdot k_j$ in (7), we get for $2 \leq s \leq r$,

$$\frac{a_{s+1}}{a_s} - \frac{n}{s} \geq \frac{a_s}{a_{s-1}} - \frac{n}{s - 1} \geq \ldots \geq \frac{a_2}{a_1} - \frac{n}{1} = \frac{2|E(G)|}{n} - n.$$

Since the last expression is at least $n(\epsilon - \frac{1}{r})$, we get

$$\frac{a_{s+1}}{a_s} \geq n \left( \epsilon - \frac{1}{r} + \frac{1}{s} \right).$$

Hence

$$\frac{(r + 1)k_{r+1}}{n} = \frac{a_{r+1}}{n} = \prod_{s=1}^{r} \frac{a_{s+1}}{a_s} = n^r \prod_{s=1}^{r} \left( \epsilon - \frac{1}{r} + \frac{1}{s} \right)$$
\[
\frac{(r + 1)k_{r+1}}{n} = \frac{a_{r+1}}{n} = \prod_{s=1}^{r} \frac{a_{s+1}}{a_s} = n^r \prod_{s=1}^{r} \left( \epsilon - \frac{1}{r} + \frac{1}{s} \right)
\]

\[
\geq \epsilon n^r \prod_{s=1}^{r-1} \left( \frac{1}{s} - \frac{1}{r} \right) = \epsilon n^r \prod_{s=1}^{r-1} \frac{r - s}{rs} = \epsilon n^r \frac{1}{r^{r-1}}.
\]