List coloring

Part 5.3
Theorem 5.7 (Alon): Let $s \geq 4$ be an even integer and

$$d > (s^2 2^{s+1})^2. \quad (1)$$

Let $G$ be a simple graph with $\delta(G) \geq d$. Then $\chi_\ell(G) > s$. In other words, $\chi_\ell(G) > \left(\frac{1}{2} - o(1)\right) \log_2(d)$. 
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Proof. Let $G = (V, E)$ be as above and $n = |V|$. Let $S = \{1, \ldots, s^2\}$.

Step 1. Construct a random $B \subseteq V$ by including each $v \in V$ into $B$ with probability $\frac{1}{\sqrt{d}}$ independently of each other. For every $v \in B$, let $L(v)$ be a random $s$-element subset of $S$. 

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**Step 1.** Construct a random $B \subseteq V$ by including each $v \in V$ into $B$ with probability $\frac{1}{\sqrt{d}}$ independently of each other. For every $v \in B$, let $L(v)$ be a random $s$-element subset of $S$.

Call $v \in V$ **good** if

(i) $v \notin B$ and

(ii) $\forall T \subseteq S$ with $|T| \geq 0.5s^2 \exists b \in B \cap N(v)$ s.t. $L(b) \subset T$. 
Claim 1: The probability that a given \( v \in V \) is not good is at most

\[
\frac{1}{\sqrt{d}} + \left( 1 - \frac{1}{\sqrt{d}} \right) \left( \frac{s^2}{2} \right) \left( 1 - \frac{1}{\sqrt{d}} \frac{s^2}{2} \left( \frac{s^2}{2} - 1 \right) \ldots \left( \frac{s^2}{2} - s + 1 \right) \right)^d.
\]

Proof of Claim 1. If \( v \notin B \) and \( T \) is a fixed subset of \( S \) with \(|T| = 0.5s^2\), then for each neighbor \( u \) of \( v \), the probability of the event that \( u \in B \) and \( L(u) \subset T \) is exactly

\[
\frac{1}{\sqrt{d}} \frac{s^2}{2} \left( \frac{s^2}{2} - 1 \right) \ldots \left( \frac{s^2}{2} - s + 1 \right) \left( \frac{s^2}{2} \left( \frac{s^2}{2} - 1 \right) \ldots \left( \frac{s^2}{2} - s + 1 \right) \right)^d.
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Claim 1: The probability that a given $v \in V$ is not good is at most

$$\frac{1}{\sqrt{d}} + \left(1 - \frac{1}{\sqrt{d}}\right) \left(\frac{s^2}{s^2/2}\right) \left(1 - \frac{1}{\sqrt{d}} \frac{s^2(s^2/2 - 1) \ldots (s^2/2 - s + 1)}{s^2(s^2 - 1) \ldots (s^2 - s + 1)}\right)^d.$$ 

Proof of Claim 1. If $v \notin B$ and $T$ is a fixed subset of $S$ with $|T| = 0.5s^2$, then for each neighbor $u$ of $v$, the probability of the event that $u \in B$ and $L(u) \subset T$ is exactly

$$\frac{1}{\sqrt{d}} \frac{s^2(s^2/2 - 1) \ldots (s^2/2 - s + 1)}{s^2(s^2 - 1) \ldots (s^2 - s + 1)}.$$ 

Hence the probability that there are no $u \in N(v)$ s.t. $u \in B$ and $L(u) \subset T$ is at most

$$\left(1 - \frac{1}{\sqrt{d}} \frac{s^2(s^2/2 - 1) \ldots (s^2/2 - s + 1)}{s^2(s^2 - 1) \ldots (s^2 - s + 1)}\right)^d.$$
Since the number of choices of $T$ is $\binom{s^2}{s^2/2}$, this yields Claim 1.
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Claim 2: \[
\frac{s^2}{2} \cdot \left( \frac{s^2 - 1}{2} \right) \cdots \left( \frac{s^2 - s + 1}{2} \right) \geq 2^{-s-1}.
\]

Proof of Claim 2. The LHS above is equal to

\[
\frac{1}{2^s} \prod_{i=0}^{s-1} \frac{s^2 - 2i}{s^2 - i} = \frac{1}{2^s} \prod_{i=0}^{s-1} \left( 1 - \frac{i}{s^2 - i} \right) \geq \frac{1}{2^s} \left( 1 - \sum_{i=0}^{s-1} \frac{i}{s^2 - i} \right) = \frac{1}{2^{s+1}}.
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Since the number of choices of $T$ is $\binom{s^2}{s^2/2}$, this yields Claim 1.

Claim 2: $\frac{s^2 (\frac{s^2}{2} - 1) \ldots (\frac{s^2}{2} - s + 1)}{s^2 (s^2 - 1) \ldots (s^2 - s + 1)} \geq 2^{-s - 1}$.

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Since for $s \geq 4$, $\binom{s^2}{s^2/2} \leq 2^{s^2 - 2}$, the probability that $v$ is not good is at most

$$\frac{1}{\sqrt{d}} + 2^{s^2 - 2} \left(1 - \frac{1}{\sqrt{d} 2^{s+1}}\right)^d \leq \frac{1}{\sqrt{d}} + 2^{s^2 - 2} \exp\{-\sqrt{d} 2^{-s-1}\}$$

$$\leq \frac{1}{\sqrt{d}} + \frac{1}{4} \left(\frac{2}{e}\right)^{s^2} \leq \frac{1}{4}.$$
Hence the expectation of the number of non-good vertices is lest than $n/4$. So by Markov’s inequality, the probability that the number of good vertices is at most $n/2$ is $<1/2$.

Similarly, since $E(|B|) = \frac{n}{\sqrt{d}}$, \( P[|B| \geq \frac{2n}{\sqrt{d}}] \leq \frac{1}{2} \).
Hence the expectation of the number of non-good vertices is less than $n/4$. So by Markov's inequality, the probability that the number of good vertices is at most $n/2$ is $< 1/2$.

Similarly, since $E(|B|) = \frac{n}{\sqrt{d}}$, $P[|B| \geq \frac{2n}{\sqrt{d}}] \leq \frac{1}{2}$.

So, with positive probability, (a) $|B| \leq \frac{2n}{\sqrt{d}}$ and (b) the number of good vertices $\geq \frac{n}{2}$.

Hence there is such a choice of $B$ and $\{L(b) : b \in B\}$. Fix it. Let $A$ be the set of good vertices.
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So, with positive probability, (a) $|B| \leq \frac{2n}{\sqrt{d}}$ and (b) # of good vertices $\geq \frac{n}{2}$.

Hence there is such a choice of $B$ and $\{L(b) : b \in B\}$. Fix it. Let $A$ be the set of good vertices.

Step 2. For every $v \in V - B$, let $L(v)$ be a random $s$-element subset of $S$.

Fix any $L$-coloring $f$ of $B$. 
Let $v \in A$. The key observation is that since $v$ is good, at least $s^2/2 + 1$ colors are used in $f(B \cap N(v))$.

So using Claim 2, the probability that $L(v) \subseteq f(B \cap N(v))$ is at least

$$\frac{(\frac{s^2}{2} + 1) \frac{s^2}{2} \ldots (\frac{s^2}{2} - s + 2)}{s^2(s^2 - 1) \ldots (s^2 - s + 1)} \geq 2^{-s-1}.$$
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By the independence of choices of $L(v)$, the probability that $f$ can be extended to each $v \in A$ is at most

$$\left(1 - \frac{1}{2^{s+1}}\right)^{|A|} \leq \left(1 - \frac{1}{2^{s+1}}\right)^{n/2} \leq \exp\{-n2^{-s-2}\}.$$
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Since there are at most $s^{|B|}$ $L$-colorings of $B$, the probability that there is an $L$-coloring of $B$ extendable to $A$ is at most
\[ s|B|e^{-n2^{-s-2}} \leq e^{(2n/\sqrt{d}) \ln s - n2^{-s-2}} = e^{\frac{n}{2^{s+2}} \left( \frac{2^{s+3} \ln s}{\sqrt{d}} - 1 \right)}. \]
\[ s|B| e^{-n2^{-s-2}} \leq e^{(2n/\sqrt{d}) \ln s - n2^{-s-2}} = e^{n \frac{2^{s+2}}{2s+2} \left( \frac{2^{s+3} \ln s}{\sqrt{d}} - 1 \right)}. \]

Since \( \sqrt{d} > s^2 2^{s+1} \),

\[ \frac{2^{s+3} \ln s}{\sqrt{d}} < \frac{4 \ln s}{s^2} < 1. \]
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Thus there is an \( s \)-uniform list assignment of vertices in \( A \) s.t. no \( L \)-coloring of \( B \) is extendable to \( A \). \( \square \)
\[ s|B| e^{-n2^{-s-2}} \leq e^{(2n/\sqrt{d}) \ln s - n2^{-s-2}} = e^{\frac{n}{2s+2}} \left( \frac{2^{s+3} \ln s - 1}{\sqrt{d}} \right). \]

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Conjecture (Alon and Krivelevich). There is \( C > 0 \) s.t. for each \( d \)-regular bipartite graph \( G \),

\[ \chi_l(G) \leq C \ln d. \]