List coloring

Part 5.2
**Lemma 5.3 (Small Pot Lemma):** If $\chi_{\ell}(G) > k$, then there is a $k$-uniform list $L$ s.t.

(a) $G$ has no $L$-coloring and
(b) for each $\alpha \in \bigcup_{v \in V(G)}$ one can assign a vertex $v_\alpha \in V(G)$ s.t. $\alpha \in L(v_\alpha)$ and all $v_\alpha$ are distinct.

In particular, there is a $k$-uniform list $L$ s.t.

$|\bigcup_{v \in V(G)} L(v)| < |V(G)|$ and $G$ has no $L$-coloring.
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In particular, there is a $k$-uniform list $L$ s.t. $|\bigcup_{v \in V(G)} L(v)| < |V(G)|$ and $G$ has no $L$-coloring.

Proof. Among all $k$-uniform lists $L$ s.t. $G$ has no $L$-coloring, choose with the smallest $|U|$ where $U = \bigcup_{v \in V(G)} L(v)$. Consider the auxiliary bigraph $H = H_L$ with parts $U$ and $V(G)$, where $\alpha v \in E(H)$ iff $\alpha \in L(v)$. 
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If $H$ has no matching covering $U$, then by Hall, there is a minimum $T \subseteq U$ s.t. $|N_H(T)| < |T|$. By the definition of $U$, $d_H(\alpha) \geq 1$ for each $\alpha \in U$. So, $|T| \geq 2$. 
Let $A = N_H(T)$. By the minimality of $T$, $|A| = |T| - 1$, and $H[T \cup A]$ has a matching $F$ covering $A$.

This matching defines an $L$-coloring of $G[A]$ (with all colors distinct). So, since $G$ is not $L$-colorable, $A \neq V(G)$. 
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The lists of all vertices in $V(G) - A$ are disjoint from $T$. Fix some $u \in V(G) - A$.
Define $L'(v) = L(v)$ for $v \in V(G) - A$ and $L'(v) = L(u)$ for $v \in A$. 
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Since $\bigcup_{v \in V(G)} L'(v) = U - T$, by the minimality of $U$, $G$ has an $L'$-coloring $F'$.
Then coloring $f$ where $f(v) = F'(v)$ for $v \in V(G) - A$ and $f(v) = F(v)$ for $v \in A$ is an $L$-coloring $G$, a contradiction. \qed
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The lemma tells us that it is enough to try a bounded number of distinct lists for a given graph.
Let $n_k$ be the minimum $s$ s.t. $\chi_{\ell}(K_s,s) \geq k + 1$.
Let $m(r,2)$ be the minimum number of edges in an $r$-uniform non-2-colorable hypergraph.
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**Theorem 5.4 (Erdős, Rubin and Taylor 1979).**

$m(k, 2) \leq n_{2k} \leq 2m(k, 2)$. 
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**Theorem 5.4 (Erdős, Rubin and Taylor 1979).**

$m(k,2) \leq n_{2k} \leq 2m(k,2)$.

**Proof.** Let $H$ be a non-2-colorable $k$-graph with $m(k,2)$ edges. Let $G = K_{m(k,2),m(k,2)}$ with parts $V_1$ and $V_2$ in which each partite set is $E(H)$. For each $e \in V(G)$, let $L(e) = \{v \in V(H) : v \in e\}$. 
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Suppose $G$ has an $L$-coloring $f$. Let $C_1 = f(V_1)$, $C_2 = f(V_2)$, and $C_0 = \bigcup_{e \in V(G)} f(e) - C_1 - C_2$.
Then in $H$ we can color $C_0 \cup C_1$ with color 1 and $C_2$ with color 2.
Suppose now that $G = K_{n_k,n_k}$ with parts $V_1$ and $V_2$, and $L$ is a $k$-uniform list for $G$ s.t. $G$ is not $L$-colorable.
Let $V(H) = \bigcup_{v \in V(G)} L(v)$ and $E(H) = \{L(v) : v \in V(G)\}$. 
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Let $V(H) = \bigcup_{v \in V(G)} L(v)$ and $E(H) = \{L(v) : v \in V(G)\}$.

If $H$ were 2-colorable, let $C_1$ and $C_2$ be the color classes. By definition, for each $v \in V(G)$, there is a $c_1(v) \in L(v) \cap C_1$ and a $c_2(v) \in L(v) \cap C_2$.

For $i \in [2]$ and $v \in V_i$, we let $f(v) = c_i(v)$. This gives an $L$-coloring of $G$, a contradiction. \qed
Proposition 5.5 (Erdős). \( m(k, 2) \geq 2^{k-1}. \)
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Proof. Let $H$ be a $k$-graph with $m < 2^{k-1}$ edges. Color every vertex with 1 with probability $1/2$ and with 2 with probability $1/2$ independently of each other. The probability of each edge to be monochromatic is $2^{1-k}$.
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Hence the expected number of monochromatic edges is $\frac{m}{2^{k-1}} < 1$. Therefore, there is a coloring with less than one monochromatic edges. 

\qed
Theorem 5.6 (Erdős). $m(k, 2) \leq 4k^2 2^k$. 

Proof. Let $|R| = 2k^2$ and $H_k$ be a random $k$-graph with vertex set $R$ obtained by placing $m = 4k^2 2^k$ random edges on $R$ independently from each other (repetitions are possible).

Claim: With positive probability, $\alpha(H_k) < k^2$.

Let $p_0$ be the probability of the event $\alpha(H_k) \geq k^2$.

Fix $S \subset R$ with $|S| = k^2$. The probability that $S$ is independent in $H_k$ is $p_1 = \left(1 - \left(\frac{k}{k^2}\right)^{2k^2}\right)^{m}$. (1)
Theorem 5.6 (Erdős). $m(k, 2) \leq 4k^22^k$.

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p_1 = \left( 1 - \frac{k^2}{2k^2} \right)^m.
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We have

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\binom{k^2}{k} \binom{2k^2}{k} = \frac{k^2 \cdot (k^2 - 1) \cdots \cdot (k^2 - k + 1)}{2k^2 \cdot (2k^2 - 1) \cdots \cdot (2k^2 - k + 1)}
\]

\[
\geq \frac{1}{2} \left( \frac{k^2 - k + 1}{2k^2 - k + 1} \right)^{k-1} = \frac{1}{2^k} \left( \frac{2k^2 - 2k + 2}{2k^2 - k + 1} \right)^{k-1}
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So

\[p_1 \leq \left( 1 - \frac{1}{2^{k+1}} \right)^m \leq \exp \left\{ -\frac{m}{2^{k+1}} \right\} \leq \exp \left\{ -\frac{4k^22^k}{2^{k+1}} \right\} = e^{-2k^2}.
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\]
Then \( p_0 \leq \binom{2k^2}{k^2} p_1 < \left( \frac{2}{e} \right)^{2k^2} \).