Edge coloring

Part 4.1
A (proper) \textit{k}-edge-coloring of a graph \(G\) is a mapping 
\(f : E(G) \to \{1, \ldots, k\}\) such that

\[f^{-1}(i) \text{ is a matching for all } i \in \{1, \ldots, k\}.\]  \hspace{1cm} (1)
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Observations: 1. If \( G \) has a loop, then it has no \( k \)-edge-coloring 
for any \( k \).
2. **Multiple edges** DO affect coloring.
3. For each \( v \in V(G) \), the colors of all incident edges are 
distinct.
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We call $f^{-1}(i)$ a color class of $f$. By definition, a $k$-edge-coloring of a graph $G$ is a partition of $E(G)$ into $k$ matchings.
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The edge chromatic number, $\chi'(G)$, of a graph $G$ is the minimum positive integer $k$ s.t. $G$ has a $k$-edge-coloring. Sometimes it is called the chromatic index of $G$. 
\( G \) is \( k \)-edge-colorable if \( \chi'(G) \leq k \).

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Fact: For each $k \geq 3$, the problem to check whether a graph $G$ with $\Delta(G) = k$ is $k$-colorable is NP-complete.
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Line graphs

For a loopless $G$, the line graph $L(G)$ has $V(L(G)) = E(G)$ and two vertices $e$ and $e'$ of $L(G)$ are adjacent iff $e$ and $e'$ share a vertex in $G$. 

By construction, $\chi'(G) = \chi(L(G))$ for every graph $G$. 
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Graph G

L(G)
It follows that $\chi'(G) \leq 2\Delta(G) - 2$ for every graph $G$. In particular, if $\Delta(G) = 3$, then $\chi'(G) \leq 4$. 

Shannon's application and example. $\Delta(S_k) = k$ and $\chi'(S_k) = \lfloor \frac{3k}{2} \rfloor$. 
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Graph $S_6$
Theorem 4.1 (Shannon’s Theorem) Let $G = (V, E)$ be a loopless graph with maximum degree $\Delta$. Then $\chi'(G) \leq 3\Delta/2$. 
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Proof. If $\Delta \leq 1$, then the statement is evident. Let $\Delta \geq 2$. We use induction on the number of edges in graphs with maximum degree at most $\Delta$. The base is the set of graphs with at most $3\Delta/2$ edges.
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Assume the theorem holds for all graphs with maximum degree at most $\Delta$ and at most $m - 1$ edges. Let $G$ have $m$ edges. Fix an edge $e_1$ in $G$. Let $v$ and $u$ be the ends of $e_1$. 
Theorem 4.1 (Shannon’s Theorem) Let $G = (V, E)$ be a loopless graph with maximum degree $\Delta$. Then $\chi'(G) \leq 3\Delta/2$.

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Let $G_1 = G - e_1$. By the minimality of $G$, graph $G_1$ has an edge coloring $f$ with colors in $M = \{1, 2, \ldots, \lfloor 3\Delta/2 \rfloor \}$.

For every $x \in V(G)$, let $O_f(x)$ be the set of colors in $M$ NOT used in $f$ to color edges incident with $v$. 
Clearly, $|O_f(x)| \geq \lfloor \Delta/2 \rfloor$ for every $x \in V(G)$. Moreover, since $e_1$ was deleted,

$$|O_f(v)| \geq \lfloor \Delta/2 \rfloor + 1, \quad |O_f(u)| \geq \lfloor \Delta/2 \rfloor + 1.$$  \hspace{1cm} (2)
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\]  (2)

The main observation is that every bicolored set of edges spans a set of vertex disjoint cycles and paths.

**Claim 1:** \( O_f(v) \cap O_f(u) = \emptyset \).

**Proof of Claim 1.** Otherwise color \( e_1 \) with any \( \alpha \in O_f(v) \cap O_f(u) \).
Claim 2: If $\alpha \in O_f(v)$ and $\beta \in O_f(u)$, then there is a $v, u$-path whose edges colored alternately with $\beta$ and $\alpha$.

Proof of Claim 2. Otherwise recolor the edges of the bicolored path of colors $\beta$ and $\alpha$ starting at $v$. 
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Proof of Claim 2. Otherwise recolor the edges of the bicolored path of colors $\beta$ and $\alpha$ starting at $v$.

Fix some $\alpha \in O_f(v), \beta \in O_f(u)$. Let $e_2$ be the edge of color $\alpha$ incident with $u$, and $w$ be the other end of $e_2$.

Claim 3: $O_f(w) \cap O_f(u) = \emptyset$.

Proof of Claim 3. If $\gamma \in O_f(w) \cap O_f(u)$, then recolor $e_2$ with $\gamma$, and color $e_1$ with $\alpha$. 
Claim 4: $O_f(w) \cap O_f(v) \neq \emptyset$.

Proof of Claim 4. By (2),

$$|O_f(w)| + |O_f(v)| + |O_f(u)| \geq \left\lfloor \frac{\Delta}{2} \right\rfloor + \left\lfloor \frac{\Delta}{2} \right\rfloor + 1 + \left\lfloor \frac{\Delta}{2} \right\rfloor + 1 > |M|.$$ 

On the other hand, by Claims 1 and 3, 

$(O_f(v) \cup O_f(w)) \cap O_f(u) = \emptyset$. This proves the claim.
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This proves the claim.

Let $\gamma \in O_f(w) \cap O_f(v)$. By Claim 2 (with $\gamma$ in place of $\alpha$), there is a $v, u$-path $P$ whose edges colored alternately with $\beta$ and $\gamma$. This path cannot go through $w$ since $\gamma \in O_f(w)$. Therefore, recoloring the edges of $P$, we come to a contradiction with Claim 3 (now $\gamma \in O(u) \cap O(w)$).
Let $\mu(G)$ denote the maximum multiplicity of an edge in $G$.

**Theorem 4.2 (Vizing, 1963)** Let $G = (V, E)$ be a multigraph with maximum degree $\Delta$. Then $\chi'(G) \leq \Delta + \mu(G)$.

Moreover, the proof yields a polynomial time algorithm of edge coloring of $G$ with $\Delta(G) + \mu(G)$ colors. But the problem of edge coloring of $G$ with $\Delta(G)$ colors is NP-complete.
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We present a proof of the following partial case of Theorem 4.2 (see the general proof in the book).

**Theorem 4.2’ (Vizing, 1963)** Let $G = (V, E)$ be a simple graph with maximum degree $\Delta$ and let $e_0 = xy \in E(G)$. If $G - e_0$ has an edge-$(\Delta + 1)$-coloring $\phi$, then $G$ also has an edge-$(\Delta + 1)$-coloring.
Suppose $G$ has no edge-$(\Delta + 1)$-coloring. Construct the auxiliary digraph $H$ as follows: $V(H) = N_G(y)$ and $uv \in E(H)$ if $\phi(vy) \in O(u)$. Let $X$ be the set of vertices reachable in $H$ from $x$ and $H'$ be the subdigraph $H[X]$ of $H$ induced by $X$. 
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By definition, $x \in X$. Since the outneighbors of a reachable from $x$ vertex also are reachable from $x$,

$$N^+_H(v) \subseteq X \quad \text{for every } v \in X. \quad (3)$$
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**Claim 1:** $\alpha \notin O(v)$ for every $v \in X$.

**Proof:** Suppose $\alpha \in O(v)$ and $(x_0, x_1, \ldots, x_s)$ where $x_0 = x$ and $x_s = v$ is an $x, v$-path in $H'$. By the definition of edges in $H'$, for each $i \in \{1, \ldots, s\}$, $\phi(yx_i) \in O(x_{i-1})$. 
Rencolor $e_s$ with $\alpha$ and for every $i = 1, \ldots, s$, (re)color $e_{i-1}$ with $\phi(e_i)$. This yields a coloring of $G$. \qed
Recolor $e_s$ with $\alpha$ and for every $i = 1, \ldots, s$, (re)color $e_{i-1}$ with $\phi(e_i)$. This yields a coloring of $G$. □

Claim 1 yields that for every $v \in X$ and $\beta \in O(v)$, there is some $w \in N(y)$ with $\phi(wy) = \beta$. Then by the definition of $H$, $vw \in E(H')$. So by (3),

$$d^+_{H'}(v) \geq \Delta + 1 - d(v) \ \forall v \in X; \ d^+_{H'}(x) \geq \Delta + 1 - d(x) + 1. \ (4)$$
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$$d_{H'}^+(v) \geq \Delta + 1 - d(v) \ \forall v \in X; \ d_{H'}^+(x) \geq \Delta + 1 - d(x) + 1. \quad (4)$$

A $[\beta, \gamma]$-path in $G$ is a path whose edges are alternately colored with $\beta$ and $\gamma$. A $[\beta, \gamma](a, b)$-path is a $[\beta, \gamma]$-path from $a$ to $b$ in $G$.

**Claim 2:** If $v \in X$ and $\beta \in O(v)$, then $G$ contains an $[\alpha, \beta](v, y)$-path.

**Proof:** If the claim is not true, choose a vertex $v \in X$ at minimum distance from $x$ in $H'$ for which there is $\beta \in O(v)$ such that the $[\alpha, \beta]$-path $P$ starting at $v$ does not end at $y$. Let $z$ denote the other end of $P$. 
Let \((x_0, x_1, \ldots, x_s)\) where \(x_0 = x\) and \(x_s = v\) be a shortest \(x, v\)-path in \(H'\). By definition, for each \(i \in \{1, \ldots, s\}\), 
\[\phi(yx_i) \in O(x_{i-1}).\]
Let \((x_0, x_1, \ldots, x_s)\) where \(x_0 = x\) and \(x_s = v\) be a shortest \(x, v\)-path in \(H'\). By definition, for each \(i \in \{1, \ldots, s\}\), 
\[\phi(yx_i) \in O(x_{i-1}).\]

If \(z \in X\), then by Claim 1 the last edge of \(P\) has color \(\alpha\), and \(\beta \in O(z)\). So by the minimality of the distance of \(v\) from \(x\), 
\(z \not\in \{x_0, \ldots, x_s\}\).

Then we can switch the colors \(\alpha\) and \(\beta\) on the edges of \(P\), recolor \(e_s\) with \(\alpha\) and for every \(i = 1, \ldots, s\) (as in the proof of Claim 1), (re)color \(yx_{i-1}\) with \(\phi(yx_i)\).
Let \((x_0, x_1, \ldots, x_s)\) where \(x_0 = x\) and \(x_s = v\) be a shortest \(x, v\)-path in \(H'\). By definition, for each \(i \in \{1, \ldots, s\}\), 
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**Claim 3:** For all distinct \(v, w \in X\), \(O(v) \cap O(w) = \emptyset\).

**Proof:** If \(v, w \in X\) and \(\beta \in O(v) \cap O(w)\), then by Claim 2, the \([\beta, \alpha]\)-path starting at \(y\) must end at both \(v\) and \(w\), an impossibility.
Claim 3 yields that

\[ d_{H'}(v) \leq 1 \text{ for every } v \in X, \text{ and } d_{H'}(x) = 0. \] (5)

Since \( \sum_{v \in X} d_{H'}(v) = |E(H')| = \sum_{v \in X} d_{H'}^+(v) \), by (4) and (5),

\[ 0 = \sum_{v \in X} (d^+_{H'}(v) - d^-_{H'}(v)) \geq 2 + \sum_{v \in X} (D + 1 - d(v) - 1) \geq 2, \] (6)

a contradiction.
Claim 3 yields that

\[ d_{H'}^-(v) \leq 1 \text{ for every } v \in X, \text{ and } d_{H'}^-(x) = 0. \] (5)

Since \( \sum_{v \in X} d_{H'}^-(v) = |E(H')| = \sum_{v \in X} d_{H'}^+(v) \), by (4) and (5),

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a contradiction. \( \square \)

A graph \( G \) is critical, if \( \chi'(G) = \Delta(G) + 1 \) and \( \chi'(G - e) < \chi'(G) \) for every \( e \in E(G) \).

Theorem 4.3 (Vizing’s Adjacency Lemma, 1965) If \( G \) is a critical graph with maximum degree \( D \geq 2 \) and \( xy \in E(G) \), then \( y \) has at least \( \max \{2, D - d(x) + 1\} \) neighbors of degree \( D \).
By criticality, graph $G' = G - xy$ has an edge-$D$-coloring $\phi$. Define $O(v)$, $H$, $X$ and $H'$ exactly as in the proof of Vizing’s Theorem. Again inequalities (3)–(5) and Claims 1–3 hold with proofs repeated word by word. Let $X' = \{ v \in X : d(v) = D \}$. Similarly to (6),

$$0 = \sum_{v \in X} (d^+_H(v) - d^-_{H'}(v)) \geq 2 + \sum_{v \in X} (D - 1 - d(v)). \quad (7)$$

A term $D - 1 - d(v)$ in the last sum is negative (and equals $-1$) iff $v \in X'$. Thus (7) yields $|X'| \geq 2$. Moreover, if $d(x) < D$ then by (7), $0 \geq 2 + (D - 1 - d(x)) - |X'|$, i.e. $|X'| \geq D - d(x) + 1$, as required.