Coloring

Part 3.3
The strong degree of a vertex \( v \) in a hypergraph \( G \) is the maximum number of edges \( e_1, \ldots, e_k \) containing \( v \) such that \( e_i \cap e_j = \{v\} \) for all \( i \neq j \).
The strong degree of a vertex $v$ in a hypergraph $G$ is the maximum number of edges $e_1, \ldots, e_k$ containing $v$ such that $e_i \cap e_j = \{v\}$ for all $i \neq j$.

For example, the degree of each vertex in $K^r_n$ is $\left(\begin{array}{c} n-1 \\ r-1 \end{array}\right)$ and strong degree is only $\left\lfloor \frac{n-1}{r-1} \right\rfloor$. 

Theorem 3.17 (Lovász): If the maximum strong degree of a hypergraph $G$ is $k$, then $\chi(G) \leq k + 1$.

Proof. Suppose $\chi(G) \geq k + 2$. We may assume that after deleting any vertex, the chromatic number is at most $k + 1$.

Fix $v \in V(G)$ and consider $G' = G - v$. By the assumption, $G'$ has a $(k + 1)$-coloring $f$ with colors $1, \ldots, k + 1$. We try to color $v$ with each color and must fail in all cases. Then for each $i \in [k + 1]$, there is an edge $e_i$ such that $f(w) = i$ for all $w \in e_i - v$. Then all sets $e_i - v$ are disjoint, and the strong degree of $v$ is at least $k + 1$. 

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For example, the degree of each vertex in $K_r^r$ is $\binom{n-1}{r-1}$ and strong degree is only $\lceil \frac{n-1}{r-1} \rceil$.

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The strong degree of a vertex \( v \) in a hypergraph \( G \) is the maximum number of edges \( e_1, \ldots, e_k \) containing \( v \) such that \( e_i \cap e_j = \{ v \} \) for all \( i \neq j \).

For example, the degree of each vertex in \( K_r^n \) is \( \binom{n-1}{r-1} \) and strong degree is only \( \lfloor \frac{n-1}{r-1} \rfloor \).

**Theorem 3.17 (Lovász):** If the maximum strong degree of a hypergraph \( G \) is \( k \), then \( \chi(G) \leq k + 1 \).

**Proof.** Suppose \( \chi(G) \geq k + 2 \). We may assume that after deleting any vertex, the chromatic number is at most \( k + 1 \).

Fix \( v \in V(G) \) and consider \( G' = G - v \). By the assumption, \( G' \) has a \( (k + 1) \)-coloring \( f \) with colors \( 1, \ldots, k + 1 \). We try to color \( v \) with each color and must fail in all cases. Then for each \( i \in [k + 1] \), there is an edge \( e_i \) such that \( f(w) = i \) for all \( w \in e_i - v \). Then all sets \( e_i - v \) are disjoint, and the strong degree of \( v \) is at least \( k + 1 \). \( \square \)
Recall:
Theorem 3.15 (Lovász, 1966 (born 1948)): Let $G$ be a graph. If $D_1, \ldots, D_t$ are nonnegative integers such that

$$\sum_{i=1}^{t} (D_i + 1) \geq \Delta(G) + 1,$$

then there is a partition $(V_1, \ldots, V_t)$ of $V(G)$ s.t.

$$\Delta(G[V_i]) \leq D_i \quad \forall i \in [t].$$
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\Delta(G[V_i]) \leq D_i \quad \forall i \in [t].
\]

Conjecture (Correa, Havet and Sereni, 2009): There exists an integer $k_0 \geq 3$ such that for each $k \geq k_0$, the vertex set of every planar graph $G$ with maximum degree at most $2k + 2$ can be partitioned into subsets $V_1$ and $V_2$ such that $\Delta(G[V_i]) \leq k$ for $i = 1, 2$. 
Theorem 3.18 (Catlin): Let $D \geq 4$. If $G$ has no 4-cycles and $\Delta(G) \leq D$, then $\chi(G) \leq 2 \left\lceil \frac{D+2}{3} \right\rceil$.

Proof. Let $t = \left\lceil \frac{D+2}{3} \right\rceil$. Let $\mathcal{P} = (V_1, \ldots, V_t)$ be a partition of $V(G)$ s. t.
(a) $\mathcal{P}$ minimizes $\sum_{i=1}^{t} |E(G[V_i])|$; and
(b) modulo (a), $\mathcal{P}$ minimizes the total number of cycles in $G[V_1] \cup \ldots \cup G[V_t]$. 
Theorem 3.18 (Catlin): Let $D \geq 4$. If $G$ has no 4-cycles and $\Delta(G) \leq D$, then $\chi(G) \leq 2 \left\lceil \frac{D+2}{3} \right\rceil$.

Proof. Let $t = \left\lceil \frac{D+2}{3} \right\rceil$. Let $P = (V_1, \ldots, V_t)$ be a partition of $V(G)$ s. t. (a) $P$ minimizes $\sum_{i=1}^{t} |E(G[V_i])|$; and (b) modulo (a), $P$ minimizes the total number of cycles in $G[V_1] \cup \ldots \cup G[V_t]$.

First we claim:

For each $P = (V_1, \ldots, V_t)$ satisfying (a),

\[ \Delta(G[V_i]) \leq 2 \text{ for every } i \in \{1, \ldots, t\}. \tag{2} \]

Indeed, suppose $d_{G[V_1]}(v) \geq 3$ for some $v \in V_1$. Since $3t \geq D + 2$, there is some $j$ such that $e(v, V_j) \leq 2$. Let $P' = (V'_1, \ldots, V'_t)$ be obtained from $P$ by moving $v$ from $V_1$ to $V_j$. Then
\[
\sum_{i=1}^{t} |E(G[V_i'])| \leq \sum_{i=1}^{t} |E(G[V_i])| - 3 + 2 < \sum_{i=1}^{t} |E(G[V_i])|,
\]

contradicting (a). This proves (2).
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contradicting (a). This proves (2).

If none of \( G[V_i] \)'s has a cycle, then each of them is 2-colorable; so \( \chi(G) \leq \sum_{i=1}^{t} \chi(G[V_i]) \leq 2t \), as claimed.

Suppose that \( G[V_{i_1}] \) has a cycle \( C_1 \). Rename \( \mathcal{P} \) as \( \mathcal{P}_0 = (V_{0,1}, \ldots, V_{0,t}) \). Let \( v_1 \in C_1 \) and \( x_1 \) and \( y_1 \) be the neighbors of \( v_1 \) in \( C_1 \). Since \( 2 + 3(t-1) = 3t - 1 \geq D + 1 \), there is \( i_2 \neq i_1 \) such that \( e(v_1, V_{0, i_2}) \leq 2 \).
\[ \sum_{i=1}^{t} |E(G[V_i])| \leq \sum_{i=1}^{t} |E(G[V_i])| - 3 + 2 < \sum_{i=1}^{t} |E(G[V_i])|, \]

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Let \( \mathcal{P}_1 = (V_{1,1}, \ldots, V_{1,t}) \) be obtained from \( \mathcal{P}_0 \) by moving \( v_1 \) from \( V_{0,i_1} \) to \( V_{0,i_2} \). Since

\[ \sum_{i=1}^{t} |E(G[V_{1,i}])| \leq \sum_{i=1}^{t} |E(G[V_{0,i}])| - 2 + 2 = \sum_{i=1}^{t} |E(G[V_{0,i}])|, \]
by (2), $\Delta(G[V_{1,i}]) \leq 2$ for every $i \in \{1, \ldots, t\}$. Thus the component $C_2$ of $V_{1,i_2}$ containing $v_1$ is either a cycle or a path, and

> the component $C_1 - v_1$ of $G[V_{1,i_1}]$ is an $x_1, y_1$-path. \hspace{1cm} (3)
by (2), $\Delta(G[V_1,i]) \leq 2$ for every $i \in \{1, \ldots, t\}$. Thus the component $C_2$ of $V_{1,i_2}$ containing $v_1$ is either a cycle or a path, and

> the component $C_1 - v_1$ of $G[V_1,i_1]$ is an $x_1, y_1$-path.  \(3\)

Moreover, by (b), $C_2$ is a cycle.

> Let $v_2 \in C_2 - v_1$ and $x_2$ and $y_2$ be the neighbors of $v_2$ in $C_2$ (possibly, $v_1 \in \{x_2, y_2\}$).

(4)

As above, there is $i_3 \neq i_2$ such that $e(v_2, V_{1,i_3}) \leq 2$ (possibly, $i_3 = i_1$).
by (2), $\Delta(G[V_1,i]) \leq 2$ for every $i \in \{1, \ldots, t\}$. Thus the component $C_2$ of $V_{1,i_2}$ containing $v_1$ is either a cycle or a path, and

$$\text{the component } C_1 - v_1 \text{ of } G[V_1,i_1] \text{ is an } x_1, y_1\text{-path.} \quad (3)$$

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As above, there is $i_3 \neq i_2$ such that $e(v_2, V_{1,i_3}) \leq 2$ (possibly, $i_3 = i_1$).

Let $\mathcal{P}_2 = (V_{2,1}, \ldots, V_{2,t})$ be obtained from $\mathcal{P}_1$ by moving $v_2$ from $V_{1,i_2}$ to $V_{1,i_3}$. Continue this process until for some $h > j \geq 0$, vertex $v_h$ is moved to $V_{h,i_{j+1}}$ and is adjacent to a vertex in $C_{j+1} - v_{j+1}$.
By (2) and (b), the component containing $v_h$ in $G[V_{h,i+1}]$ is a cycle, and thus by (3) with $j + 1$ in place of 1, $v_h$ must be adjacent to $x_{j+1}$ and $y_{j+1}$. But then we have the 4-cycle $(v_h, x_{j+1}, v_{j+1}, y_{j+1})$, a contradiction. \qed
By (2) and (b), the component containing $v_h$ in $G[V_{h,i+1}]$ is a cycle, and thus by (3) with $j+1$ in place of 1, $v_h$ must be adjacent to $x_{j+1}$ and $y_{j+1}$. But then we have the 4-cycle $(v_h, x_{j+1}, v_{j+1}, y_{j+1})$, a contradiction. □

Theorem 3.19 (A. K.): Let $D \geq 4$. If $G$ has no 3-cycles and $\Delta(G) \leq D$, then $\chi(G) \leq 2 \left\lceil \frac{D+2}{3} \right\rceil$.

Proof. We repeat the proof of Theorem 3.18 until (4). In (4), since $G$ is triangle-free, we may additionally demand that $v_2 v_1 \notin E(G)$, i.e., $v_1 \notin \{x_2, y_2\}$. Then we continue the proof until the last sentence, demanding for each $j$ that $v_j \notin \{x_{j+1}, y_{j+1}\}$. Instead of the last sentence of the proof of Theorem 3.18, we argue as follows.
Since $G$ is triangle-free, $v_h v_{j+1} \notin E(G)$. Let $W$ be the set of neighbors of $v_{j+1}$ in $V_{h,i_{j+1}} - \{x_{j+1}, y_{j+1}\}$. Since the only neighbors of $v_{j+1}$ in $V_{j,i_{j+1}}$ were $x_{j+1}$ and $y_{j+1}$, every vertex in $W$ is of the form $v_g$ for some $j + 2 \leq g \leq h - 1$. Hence each $w \in W$ has exactly 2 neighbors in $V_{h,i_{j+1}}$, and $W$ is independent.
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Therefore, for each $w \in W$ there is an $i(w) \neq i_{j+1}$ such that $w$ has at most 2 neighbors in $V_{h,i(w)}$. Let partition $Q = (V'_1, \ldots, V'_t)$ be obtained from $P_h$ by moving $v_{j+1}$ into $V_{h,i_{j+1}}$ and each $w \in W$ into $V_{h,i(w)}$. 
Since $G$ is triangle-free, $v_h v_{j+1} \notin E(G)$. Let $W$ be the set of neighbors of $v_{j+1}$ in $V_{h,i_{j+1}} - \{x_{j+1}, y_{j+1}\}$. Since the only neighbors of $v_{j+1}$ in $V_{j,i_{j+1}}$ were $x_{j+1}$ and $y_{j+1}$, every vertex in $W$ is of the form $v_g$ for some $j + 2 \leq g \leq h - 1$. Hence each $w \in W$ has exactly 2 neighbors in $V_{h,i_{j+1}}$, and $W$ is independent.

Therefore, for each $w \in W$ there is an $i(w) \neq i_{j+1}$ such that $w$ has at most 2 neighbors in $V_{h,i(w)}$. Let partition $Q = (V'_1, \ldots, V'_t)$ be obtained from $P_h$ by moving $v_{j+1}$ into $V_{h,i_{j+1}}$ and each $w \in W$ into $V_{h,i(w)}$.

By the choice of $i(w)$ and the fact that $W$ is independent, $\sum_{i=1}^t |E(G[V'_i])| \leq \sum_{i=1}^t |E(G[V_{h,i}])|$. But the graph $G[V'_{i_{j+1}}]$ has now vertices $x_{j+1}$ and $y_{j+1}$ of degree 3, a contradiction to (2). \qed
Better bounds

**Theorem 3.20 (Johansson, 1996)** If $G$ is a triangle-free graph with $\Delta(G) \leq D$, then

$$\chi(G) \leq \frac{9D + o(D)}{\ln D}.$$
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1. Kim, Molloy, Bernshteyn.
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2. It is known that for each $g$ and each $D$ there are graphs $G(g, D)$ with maximum degree $D$, girth $g$ and chromatic number at least $\frac{D}{2\ln D}$. 
Better bounds

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3. Discussion of small \( D \). Unsolved problems.