Coloring

Part 3.1
Definitions

A (proper) $k$-coloring of the vertices of a graph $G$ is a mapping $f : V(G) \to \{1, \ldots, k\}$ such that

$$f(x) \neq f(y) \quad \forall \ e \in E(G) \text{ with ends } x \text{ and } y. \quad (1)$$
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$$f(x) \neq f(y) \quad \forall \ e \in E(G) \text{ with ends } x \text{ and } y.$$  \ (1)

Observations: 1. If $G$ has a loop, then it has no $k$-coloring for any $k$.
2. Multiple edges do not affect coloring. So below we consider colorings only simple graphs.
Observation: Given a $k$-coloring $f$ of the vertices of a graph $G$, for each $i \in \{1, \ldots, k\}$, $f^{-1}(i)$ is an independent set. We call $f^{-1}(i)$ a color class of $f$. So, a $k$-coloring of the vertices of a graph $G$ is a partition of $V(G)$ into $k$ independent sets.
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The larger is $k$, the more freedom we have. We always can color the vertices of an $n$-vertex graphs with $n$ colors. The chromatic number, $\chi(G)$, of a graph $G$ is the minimum positive integer $k$ s.t. $G$ has a $k$-coloring.

$G$ is $k$-colorable if $\chi(G) \leq k$. 
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Question: Which graphs are 2-colorable?
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Question: Which graphs are 2-colorable?

Fact: For each $k \geq 3$, the problem to check whether a graph $G$ is $k$-colorable is NP-complete.

The clique number, $\omega(G)$, is the size of a largest clique (complete subgraph) in $G$.

**Proposition 3.1.** For every graph $G$, $\chi(G) \geq \omega(G)$ and $\chi(G) \geq |V(G)|/\alpha(G)$.

**Proof.** All vertices in a clique of size $\omega(G)$ must have different colors. This proves $\chi(G) \geq \omega(G)$.

With any color, we can color at most $\alpha(G)$ vertices. This proves $\chi(G) \geq |V(G)|/\alpha(G)$.

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Greedy coloring

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2. For $i = 1, 2, \ldots, n$ color $v_i$ with the smallest positive integer distinct from the colors of the neighbors $v_j$ of $v_i$ with $j < i$. 

Proposition 3.2. For every graph $G$, $\chi(G) \leq 1 + \Delta(G)$.

Proof. Apply greedy coloring to $G$. At every step, at most $\Delta(G)$ colors are forbidden for $v_i$. So, there always is a color in $\{1, \ldots, 1 + \Delta(G)\}$ available to color $v_i$.

On the other hand, on the next slide we will see an example of a tree $T_4$ with an ordering of its vertices s.t. the greedy coloring of $T_4$ w.r.t. this ordering needs 4 colors. It is clear how to generalize this to a tree that will need a 1000 colors for its greedy coloring.
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Definition A. A graph $G$ is $d$-degenerate if for every subgraph $H$ of $G$, $\delta(H) \leq d$.

Example 1: A graph $G$ is 1-degenerate iff $G$ is a forest.

Example 2: Every planar graph is 5-degenerate.

Definition B. A graph $G$ is $d$-degenerate if its vertices can be ordered $v_1, \ldots, v_n$ so that for each $1 < i \leq n$, vertex $v_i$ has at most $d$ neighbors in $\{v_1, \ldots, v_{i-1}\}$.

Proposition 3.3. Definitions (A) and (B) are equivalent.
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Proposition 3.3. Definitions (A) and (B) are equivalent.

Proof: In class.

Proposition 3.4. Every $d$-degenerate graph is $(d + 1)$-colorable.

Proof: Use the ordering of vertices provided by Definition B, and apply to it the greedy coloring.
Orderings and coloring

Theorem 3.5 (Minty, 1962): Let $G$ be a graph with a nonempty set $C$ of cycles. For an orientation $D$ of $G$, let

$$r(D) = \max_{C \in C} \left\lfloor \frac{a}{b} \right\rfloor,$$

where $a$ and $b$ are the number of edges of $C$ oriented clockwise and counterclockwise, respectively. Then $\chi(G) = 1 + \min r(D)$, where the minimum is taken over all orientations $D$ of $G$. 
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Proof. Part $\geq$. Let $f$ be an optimal coloring with colors 1, $\ldots$, $k$. Construct $D^*$: Orient $ab$ if $f(a) < f(b)$. Then each dipath in $D^*$ has $\leq k - 1$ edges. So $r(D^*) \leq k - 1 = \chi(G) - 1$. 

Proof. Part $\leq$. May assume $G$ is connected. Also, it is enough to consider acyclic orientations. Let $D$ be such an orientation.
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Part $\leq$. May assume $G$ is connected. Also, it is enough to consider acyclic orientations. Let $D$ be such an orientation.
Fix $x \in V(G)$. For a walk $W$ starting from $x$, let

$$g(W) = a - b \cdot r(D),$$

where $a$ is the number of the edges in $W$ oriented forward, and $b$ – backward.

For $y \in V(G)$, let

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By definition, $g(W) \leq 0$ for each cycle $W$. Hence $g(y)$ is attained at some $x, y$-path. So, $g(y)$ is well defined.

If $uv \in E(D)$, then $g(v) \geq 1 + g(u)$ and $g(u) \geq g(v) - r(D)$. 
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If \( uv \in E(D) \), then \( g(v) \geq 1 + g(u) \) and \( g(u) \geq g(v) - r(D) \).

So we can color \( V(G) \) by congruence classes of \( g \) modulo \( 1 + r(D) \).
Theorem 3.6 (Vitaver 1962, Hasse 1964/65, Roy 1967, Gallai 1968): If $D$ is an orientation of $G$, then

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where $\ell(D)$ is the length of a longest path in $D$. Moreover, there is $D^*$ with equality in (3).
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Proof. Equality proved as Part $\geq$ in Theorem 3.5.

Let $D$ be any orientation of $G$ and $\ell = \ell(D)$. Let $D'$ be an acyclic subgraph of $D$ with the most edges. Let $D''$ be obtained from $D$ by reversing all edges in $E(D) - E(D')$. Then

(a) $D''$ is acyclic and (b) $\ell(D'') \leq \ell(D)$.

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(a) \( D'' \) is acyclic and (b) \( \ell(D'') \leq \ell(D) \).

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Since \( D'' \) is acyclic, apply Theorem 3.5 to it.
Theorem 3.7 (Brooks, 1941): If $\Delta(G) = k$ and $\chi(G) > k$, then either $G$ contains $K_{k+1}$ or $k = 2$ and $G$ contains an odd cycle.
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Proof (Mel’nikov and Vizing): Case $k \leq 2$ is easy. Let $k \geq 3$ and $G$ be a minimum counter-example. Let $v \in V(G)$, $G' = G - v$, $f$ be a $k$-coloring of $G$, and $V_1, \ldots, V_k$ be the color classes of $f$. 
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Let $G_{i,j} = G'[V_i \cup V_j]$ and $G_{i,j}(x)$ denote the component of $G_{i,j}$ containing $x$.

Then $v$ has exactly one neighbor, say $x_i$, in each $V_i$.

Claim 1: $\forall 1 \leq i < j \leq k$, $G_{i,j}(x_i) = G_{i,j}(x_j)$.

Claim 2: $\forall 1 \leq i < j \leq k$, $G_{i,j}(x_i)$ is an $x_i, x_j$-path.

Claim 3: For all distinct $i, j, s$, $G_{i,j}(x_i)$ and $G_{i,s}(x_i)$ share only $x_i$. 
Finishing proof. Choose a $k$-coloring $f$ of $G'$ so that to maximize $G_{1,2}(x_1)$.

By Claim 2, it is an $x_1, x_2$-path, say $y_1 y_2 \ldots y_q$ where $y_1 = x_1$, $y_q = x_2$. 
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If $q = 2$, then $G = K_{k+1}$. So $q \geq 4$ and $y_{q-1} \neq x_1$.

Recolor $G_{2,3}(x_2)$ and call the new coloring $f^*$. 
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Consider $G_{1,2}^*$. By Claim 3, $G_{1,2}^* \supseteq G_{1,2} - x - 2$. By the maximality of $G_{1,2}(x_1)$, $y_{q-1} x_3 \in E(G)$. But then

$$y_{q-1} \in G_{1,3}(x_3) = G_{1,3}(x_1),$$

contradicting Claim 3. \qed
Corollary 3.8 of the proof (A.K. and Nakprasit): Suppose $\Delta(G) \leq k \geq 3$ and $G$ does not contain $K_{k+1}$. If for some $v \in V(G)$ $G - v$ has a $k$-coloring $f$ with color classes $M_1, \ldots, M_k$, then $G$ has a $k$-coloring $f'$ with color classes $M'_1, \ldots, M'_k$ s.t. $|M'_i| = M_i$ for all $i \in [k]$ apart from one.

**Proof.** Essentially, repeats the proof of Th. 3.7. Different proof of Claim 1 (combined with Claim 2).
Corollary 3.8 of the proof (A.K. and Nakprasit): Suppose \(\Delta(G) \leq k \geq 3\) and \(G\) does not contain \(K_{k+1}\). If for some \(v \in V(G)\) \(G - v\) has a \(k\)-coloring \(f\) with color classes \(M_1, \ldots, M_k\), then \(G\) has a \(k\)-coloring \(f'\) with color classes \(M'_1, \ldots, M'_k\) s.t. \(|M'_i| = M_i\) for all \(i \in [k]\) apart from one.

**Proof.** Essentially, repeats the proof of Th. 3.7. Different proof of Claim 1 (combined with Claim 2).

Corollary 3.9: Suppose \(\Delta(G) \leq k \geq 3\) and \(G\) does not contain \(K_{k+1}\). If for some \(1 \leq s < k\) and a positive integer \(m\), \(G\) contains an \(s\)-colorable induced subgraph \(G'\) with \(m\) vertices, then \(G\) has a \(k\)-coloring in which the union of some \(s\) color classes is at least \(m\). In particular, \(G\) has a \(k\)-coloring in which a color class has \(\alpha(G)\) vertices.