Lemma 1 Let $G$ be a connected loopless graph with weighted edges, where $w(e) \geq 0$ for every $e \in E(G)$. Let $T_1, \ldots, T_k$ be vertex-disjoint trees contained in $G$ such that $V(T_1) \cup \ldots \cup V(T_k) = V(G)$. Let $e_0$ be an edge of a minimum weight among the edges of $G$ connecting $V(T_1)$ with $V(G) - V(T_1)$. Then among the spanning trees of $G$ containing $E(T_1) \cup \ldots \cup E(T_k)$ of minimum weight, there is a tree containing $e_0$.

PROOF. Let $n = V(G)$. Let $T_0$ be a spanning tree of $G$ containing $E(T_1) \cup \ldots \cup E(T_k)$ of minimum weight. Suppose $e_0 = xy$ where $x \in V(T_1)$ and $y \in V(G) - V(T_1)$. If $e_0 \in E(T_0)$, then we are done. Otherwise, $T' = T_0 + e_0$ is a connected graph with $n$ edges containing exactly one cycle, say $C$. By construction, $e_0 \in E(C)$. Since $x \in V(T_1)$ and $y \in V(G) - V(T_1)$, cycle $C$ contains another edge $e_1$ connecting $V(T_1)$ with $V(G) - V(T_1)$. Then $T'' := T' - e_1$ is a connected graph with $n-1$ edges; hence a spanning tree of $G$. Moreover, by the choice of $e_0$, $w(e_0) \leq w(e_1)$.

Therefore, $\sum_{e \in E(T')} w(e) \leq \sum_{e \in E(T_0)} w(e)$. It follows that $T''$ also is a spanning tree of $G$ containing $E(T_1) \cup \ldots \cup E(T_k)$ of minimum weight. This proves the lemma.

Prim’s Algorithm:
Input: A weighted connected $n$-vertex graph $G$, say, $V(G) = \{v_1, \ldots, v_n\}$.
Goal: A spanning tree with the minimum total weight of the edges.
Initialization: Let $V_0 := \{v_1\}$ and $E(T) := \emptyset$.
Step $i$ ($i = 1, \ldots, n-1$): Let $e_i$ be an edge of minimum weight among the edges connecting $V_0$ with $V(G) - V_0$. If $e_i = xy$ where $x \in V_0$ and $y \notin V_0$, then let $V_0 := V_0 \cup \{y\}$, $E(T) := E(T) \cup \{e_i\}$.
Proof: By Lemma 1.

Kruskal’s Algorithm:
Input: A weighted connected $m$-edge graph $G$, say, $E(G) = \{e_1, \ldots, e_m\}$.
Goal: A spanning tree with the minimum total weight of the edges.
Initialization: Reorder the edges so that $w(e_1) \leq w(e_2) \leq \ldots \leq w(e_m)$. Let $E(T) := \emptyset$.
Step $j$ ($j = 1, \ldots, m$): If $E(T) \cup \{e_j\}$ does not contain cycles, then let $E(T) := E(T) \cup \{e_j\}$.
Proof: By Lemma 1.