Theorem 1 (Mantel, 1907). Let $f(n)$ be the maximum number of edges in a simple $n$-vertex graph with no triangles. Then $f(n) = \left\lfloor \frac{n^2}{4} \right\rfloor$.

Proof. The fact that $f(n) \geq \left\lfloor \frac{n^2}{4} \right\rfloor$ follows from the example of the complete bipartite graph $K_{\lfloor n/2 \rfloor, \lceil n/2 \rceil}$, since it is bipartite (and thus has no 3-cycles) and has exactly $\left\lfloor \frac{n^2}{4} \right\rfloor$ edges.

We prove the inequality

$$f(n) \leq \left\lfloor \frac{n^2}{4} \right\rfloor$$

by induction on $n$. The only 1-vertex simple graph is $K_1$ and $|E(K_1)| = 0 = \left\lfloor \frac{1^2}{4} \right\rfloor$. Similarly, $K_2$ has the most edges among the two simple 2-vertex graphs, and $|E(K_1)| = 1 = \left\lfloor \frac{2^2}{4} \right\rfloor$.

Suppose now that (1) holds for all $n \leq k - 1$ and let $G$ be a simple $k$-vertex graph with no 3-cycles and $|E(G)| = f(k)$. Let $xy$ be any fixed edge in $G$. Since $G$ has no 3-cycles, every $z \in V(G) - x - y$ is adjacent to at most one of $x$ and $y$. Thus

$$\text{(2) the total number of edges incident with } x \text{ or } y \text{, or both is at most } k - 1.$$ 

Let $G' = G - x - y$. By construction, $G'$ is a simple $(k - 2)$-vertex triangle-free graph. By the induction assumption, $|E(G')| \leq f(k - 2) \leq \left\lfloor \frac{(k - 2)^2}{4} \right\rfloor$. But then by (2),

$$|E(G)| \leq |E(G')| + k - 1 \leq \left\lfloor \frac{(k - 2)^2}{4} \right\rfloor + k - 1 = \left\lfloor \frac{(k - 2)^2 + 4(k - 1)}{4} \right\rfloor = \left\lfloor \frac{k^2}{4} \right\rfloor,$$

as claimed. This proves the induction step and thus the theorem.