LIST STAR EDGE COLORING OF SUBCUBIC GRAPHS

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Abstract

A star edge-coloring of a graph G is a proper edge coloring such that
every 2-colored connected subgraph of G is a path of length at most 3. For
a graph G, let the list star chromatic index of G, ch′st(G), be the minimum
k such that for any k-uniform list assignment L for the set of edges, G has
a star edge-coloring from L. Dvořák, Mohar and Šámal asked whether the
list star chromatic index of every subcubic graph is at most 7. We prove
that it is at most 8. We also prove that if the maximum average degree
of a subcubic graph G is less than ⁹⁄₄ (resp., ⁷⁄₄), then ch′st(G) ≤ 5 (resp.,
ch′st(G) ≤ 6).

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1. Introduction

All the graphs we consider are finite and simple. For a graph $G$, we denote by $V(G)$, $E(G)$, $\delta(G)$ and $\Delta(G)$ its vertex set, edge set, minimum degree and maximum degree, respectively.

A proper vertex (respectively, edge) coloring of $G$ is an assignment of colors to the vertices (respectively, edges) of $G$ such that no two adjacent vertices (respectively, edges) receive the same color. A star coloring of $G$ is a proper vertex coloring of $G$ such that the union of any two color classes induces a star forest in $G$, i.e. every component of this union is a star. This notion was first mentioned by Grünbaum [6] in 1973, but attracted more attention only in 2001 after the paper [5] by Fertin, Raspaud and Reed. By now, there are more than 30 publications on this topic. The star coloring even in the class of line graphs seems to be difficult. A convenient language for discussions of star coloring of line graphs is the language of star edge-coloring of all graphs.

A star edge-coloring of a graph $G$ is a proper edge-coloring such that every 2-colored connected subgraph of $G$ is a path of length at most 3. In other words, we forbid bi-colored 4-cycles and 4-paths in $G$ (by a $k$-path we mean a path with $k$ edges). This notion is intermediate between acyclic edge-coloring, when every 2-colored subgraph must be only acyclic, and strong edge-coloring, when every 2-colored connected subgraph has at most two edges. The star chromatic index of $G$, denoted by $\chi'_{st}(G)$, is the minimum number of colors needed for a star edge-coloring of $G$. It was first studied by Liu and Deng [9] in 2008. They proved the following upper bound.

**Theorem 1.** [9] For every $G$ with maximum degree $\Delta \geq 7$, $\chi'_{st}(G) \leq \lceil 16(\Delta - 1)^{\frac{3}{2}} \rceil$

In [3] and later [2] it is proved:

**Theorem 2.** [3, 2] The star chromatic index of any tree with maximum degree $\Delta$ is at most $\Delta + \lceil \frac{\Delta - 1}{2} \rceil$.

In a seminal paper [4], Dvořák, Mohar and Šámal showed that even determining the star chromatic index of the complete graph $K_n$ with $n$ vertices is a hard problem. They gave the following bounds:

$$2n(1 + o(1)) \leq \chi'_{st}(K_n) \leq n \frac{2\sqrt{2}(1+o(1))\sqrt{\log n}}{\log n^4}.$$  

They also studied the star chromatic index of subcubic graphs, that is, graphs with maximum degree at most 3. They proved that $\chi'_{st}(G) \leq 7$ for every subcubic graph $G$, and conjectured that $\chi'_{st}(G) \leq 6$ for every such $G$.

A natural generalization of star edge-coloring is the list star edge-coloring. An edge list $L$ for a graph $G$ is a mapping that assigns a finite set of colors to
each edge of $G$. Given an edge list $L$ for a graph $G$, we say that $G$ is $L$-star
edge-colorable if it has a star edge-coloring $c$ such that $c(e) \in L(e)$ for every edge
of $G$. The list star chromatic index, $\chi_{st}^\prime(G)$, of a graph $G$ is the minimum $k$ such
that for every edge list $L$ for $G$ with $|L(e)| = k$ for every $e \in E(G)$, $G$ is $L$-star
dge-colorable.

Dvořák, Mohar and Šámal \cite[Question 3]{DMS18} asked whether $\chi_{st}^\prime(G) \leq 7$
for every subcubic $G$. We prove the following result toward this question.

**Theorem 3.** For every subcubic graph $G$, $\chi_{st}^\prime(G) \leq 8$.

![Figure 1. Two subcubic graphs with mad = 2 and list star chromatic index 5.](image)

We also give sufficient conditions for the list star chromatic index of a subcubic
graph to be at most 5 and 6 in terms of the maximum average degree $\text{mad}(G) = \max \left\{ \frac{2|E(H)|}{|V(H)|} : H \subseteq G \right\}$. Note that the best possible sufficient condition for 4
colors is $\text{mad}(G) < 2$. If $\text{mad}(G) < 2$ then $G$ is acyclic and by Theorem 2
for $\Delta = 3$, we have $\chi_{st}^\prime(G) \leq 4$. The same proof yields also $\chi_{st}^\prime(G) \leq 4$.
On the other hand, each of the graphs $G_i$ in Figure 1 has $\text{mad}(G_i) = 2$ and
$\chi_{st}^\prime(G_i) \geq \chi_{st}^\prime(G_i) = 5$. Our second result is:

**Theorem 4.** Let $G$ be a subcubic graph.

1. If $\text{mad}(G) < \frac{7}{2}$ then, $\chi_{st}^\prime(G) \leq 5$.
2. If $\text{mad}(G) < \frac{5}{2}$ then, $\chi_{st}^\prime(G) \leq 6$.

As every planar graph with girth $g$ satisfies $\text{mad}(G) < \frac{2g}{g-2}$, Theorem 4 yields
the following.

**Corollary 1.** Let $G$ be a planar subcubic graph with girth $g$.

1. If $g \geq 14$ then $\chi_{st}^\prime(G) \leq 5$. 
2. If $g \geq 10$ then $\text{ch}_{st}^e(G) \leq 6$.

Analogous to Theorem 4 bounds were earlier proved in [7] for the strong chromatic index, $\chi'_s(G)$ — the minimum $k$ such that $G$ has a strong edge-coloring with $k$ colors. Recall that a strong edge-coloring of a graph $G$ is a proper edge-coloring such that any two edges adjacent to a common edge receive different colors. Since every strong edge-coloring is also a star edge-coloring, the following results give bounds for the star chromatic index. Note that the restrictions on $\text{mad}$ in the first two statements of Theorem 5 below are the same as in Theorem 4, but the bounds are different.

**Theorem 5** [7]. Let $G$ be a subcubic graph.

1. If $\text{mad}(G) < \frac{7}{3}$ then, $\chi'_s(G) \leq 6$.

2. If $\text{mad}(G) < \frac{5}{2}$ then, $\chi'_s(G) \leq 7$.

3. If $\text{mad}(G) < \frac{8}{3}$ then, $\chi'_s(G) \leq 8$.

4. If $\text{mad}(G) < \frac{20}{7}$ then, $\chi'_s(G) \leq 9$.

List versions of two results of the previous theorem (for $\text{mad}(G) < \frac{5}{2}$ and $\text{mad}(G) < \frac{8}{3}$) are proved in [10].

The structure of the paper is as follows. In the next section we introduce some notation and prove an analog of Lemma 5.2 in [4] on extensions of partial star edge-colorings. In Section 3 we prove Theorem 3, and in the two last sections we prove Parts 1 and 2 of Theorem 4.

## 2. Preliminaries

For a graph $G$, let $d_G(v)$ denote the degree of a vertex $v$ in $G$ and $N_G(v)$ denote the set of neighbors of $v$ in $G$. If $G$ is clear from the content, we may omit the subscript. A vertex of degree $k$ is called a $k$-vertex, and a $k$-neighbor of a vertex $v$ is a $k$-vertex adjacent to $v$.

An edge $xy$ is weak if at least one of $x$ and $y$ is a leaf. A vertex $x$ is weak if at least one of the edges incident with $x$ is weak.

For brevity, we often will write "$k$-se-coloring" instead of "star edge $k$-coloring" and "se-coloring" instead of "star edge-coloring". A partial edge-coloring of a graph $G$ is an edge-coloring of a subgraph $G'$ of $G$ (where $G'$ can equal $G$).

For a partial edge-coloring $\phi$ of a graph $G$ and a vertex $v \in V(G)$, $\phi(v)$ denotes the set of colors used on the edges incident with $v$.

We will heavily use the following lemma.
Lemma 6. Let $\phi$ be a partial se-coloring of a graph $G$ and $uv$ be an uncolored edge. If $\alpha$ is a color satisfying at least one of the two properties below, then the coloring $\phi'$ obtained from $\phi$ by coloring $uv$ with $\alpha$ also is a partial se-coloring of $G$.

(a) For every $x \in N(v) \cup N(u)$, $\alpha \notin \phi(x)$;
(b) $\phi(u) \cap \phi(v) = \emptyset$, $\alpha \notin \phi(u) \cup \phi(v)$, and among the edges incident with the neighbors of $v$ or $u$, only weak edges may have color $\alpha$.

Proof. Suppose (a) or (b) holds, but $\phi'$ is not a partial se-coloring of $G$. Then there is a color $\beta$ and either a path $z_1 \ldots z_4 z_5$ or a cycle $\ldots z_4 z_1$ containing edge $uv$ whose edges are colored with $\alpha$ and $\beta$. By symmetry, we may assume that $u = z_i$ and $v = z_{i+1}$ for $i \in \{1, 2\}$. Then $\phi(z_{i+2}z_{i+3}) = \alpha$. So, (a) cannot hold. Thus (b) holds. If $i = 2$, then we have a contradiction to $\phi(u) \cap \phi(v) = \emptyset$.
So $i = 1$. But $z_3 z_4$ is not weak, which violates (b).

3. Proof of Theorem 3

Let $G$ be a subcubic graph with the minimum total number of edges and vertices such that there exists a list $L$ for the set of the edges of $G$ with $|L(e)| = 8$ for every $e \in E(G)$ for which $G$ has no $L$-star-edge-coloring.

Clearly, $G$ is connected.

Lemma 7. $G$ is 3-regular.

Proof. If $G$ has a 1-vertex $u$ adjacent to some $v$, then by the minimality of $G$, graph $G-u$ has an se-coloring $\phi$ from $L$. We view it as a partial se-coloring of $G$. Let $W$ be the set of neighbors of $v$ distinct from $u$. We extend $\phi$ by coloring $uv$ with any color $\alpha \in L(uv)$ distinct from the colors of the (at most six) edges incident with the vertices in $W$. So, $\delta(G) \geq 2$.

Suppose now that $G$ has a 2-vertex $v$ adjacent to $u$ and $w$. Let $N(u) \subseteq \{v, u_1, u_2\}$ and $N(w) \subseteq \{v, w_1, w_2\}$. By the minimality of $G$, graph $G - v$ has an $L$-coloring $\phi$ of its edges. We view it as a partial se-coloring of $G$. Let $A(uv) = L(uv) - \phi(u_1) - \phi(u_2)$ and $A(uw) = L(uw) - \phi(w_1) - \phi(w_2)$. By definition, $|A(uv)| \geq 2$ and $|A(uw)| \geq 2$. If there is $\alpha \in A(uv) - \phi(w)$, then by coloring $vw$ with some $\beta \in A(vw) - \alpha$ and $uv$ with $\alpha$ we get an se-coloring of $G$. Indeed, at each step the conditions of Lemma 6(a) will hold. Otherwise, $d(u) = d(w) = 3, d(u_1) = d(u_2) = d(w_1) = d(w_2) = 3, uw \notin E(G)$,

$$
L(uv) = \{\phi(uw_1), \phi(uw_2)\} \cup \phi(u_1) \cup \phi(u_2) \quad \text{and} \quad L(vw) = \{\phi(ww_1), \phi(ww_2)\} \cup \phi(w_1) \cup \phi(w_2).
$$

In particular, for $i = 1, 2$, vertex $u_i$ (respectively, $w_i$) has two neighbors $u'_i$ and $w'_i$ (respectively, $w''_i$ and $w'''_i$) distinct from $u$ (respectively, $w$). We then try
to color $vw$ with $\phi(uu_2)$ and $uv$ with either $\phi(u_1u'_1)$ or $\phi(u_1u''_1)$. If we do not get an se-coloring of $G$, then any 2-colored 4-path in $G$ contains edges $uv$ and $uu_1$, so that each of $u'_1$ and $u''_1$ is incident with an edge of color $\phi(uu_1)$. It follows that $|\phi(u'_1) \cup \phi(u''_1)| \leq 5$. Similarly, each of $u'_2$ and $u''_2$ is incident with an edge of color $\phi(uu_2)$, and $|\phi(u'_2) \cup \phi(u''_2)| \leq 5$. If there is $\gamma_1 \in L(uu_1) - (\phi(u'_1) \cup \phi(u''_1) \cup \phi(uu_2))$, then we color $vw$ with $\phi(uu_1)$, $vw$ with $\phi(uu_2)$, and recolor $uu_1$ with $\gamma_1$. By (1) and the definition of $\gamma_1$ this would yield an se-coloring of $G$ from $L$, a contradiction. This means

$$L(uu_1) = \phi(u'_1) \cup \phi(u''_1) \cup \phi(uu_2).$$

(2)

Similarly, $L(uu_2) = \phi(u'_2) \cup \phi(u''_2) \cup \phi(uu_1)$. In particular, $\phi(uu_2) \in L(uu_1)$ and $\phi(uu_1) \in L(uu_2)$. Then switching the colors of $uu_1$ and $uu_2$ we obtain another se-coloring $\phi'$ of $G - v$. Repeating the above argument for $\phi'$ in place of $\phi$, we get that each of $u'_1$ and $u''_1$ is incident with an edge of color $\phi'(uu_1) = \phi(uu_2)$. But then $|\phi(u'_1) \cup \phi(u''_1)| = 4$, a contradiction to (2).

In the following we will say that two edges are at distance at most 1 if they are adjacent or adjacent to a same edge.

Let $C = (v_1, \ldots, v_t)$ be a shortest cycle in $G$. Since $C$ is shortest, it has no chords. Thus for each $i = 1, \ldots, t$, vertex $v_i$ has a unique neighbor $v'_i$ in $V(G) - V(C)$. Let $G_1 = G - E(C)$. An se-coloring $\phi$ of $G_1$ from $L$ is stable if for every $i = 1, \ldots, t$, $\phi(v_iv'_i)$ differs from $\phi(v_{i-1}v'_{i-1})$, $\phi(v_{i+1}v'_{i+1})$, and from the color of each edge in $G_1$ at distance at most 1 from $v_iv'_i$ in $G_1$ (note that $G_1$ has at most six such edges: two incident with $v'_i$ and at most four others incident with the neighbors of $v'_i$).

**Lemma 8.** $G_1$ does not have stable se-colorings from $L$.

**Proof.** Suppose $G_1$ has a stable se-coloring $\phi$ from $L$. For every $i = 1, \ldots, t$, let $L'(v_iv_{i+1}) = L(v_iv_{i+1}) - \{\phi(v_{i-1}v'_{i-1}), \phi(v_iv'_i), \phi(v_{i+1}v'_{i+1}), \phi(v_{i+2}v'_{i+2})\}$ (indices taken modulo $t$).

Then $|L'(v_iv_{i+1})| \geq 4$ for every $i = 1, \ldots, t$. It is known that every cycle has an se-coloring from any 4-uniform list. (Simply, the square of any cycle of length $t \neq 5$ has a list 4-coloring, and if $t = 5$, then we can color two nonadjacent edges with one color, say $\alpha$, and all other 3 edges with different colors distinct from $\alpha$.

So, let $\phi'$ be an se-coloring of $C$ from $L'$. We claim that $\phi \cup \phi'$ is an se-coloring of $G$ from $L$. This follows from the fact that, by the definition of stable colorings and of $L'$, for every $i = 1, \ldots, t$, $\phi(v_iv'_i)$ differs from the colors of all edges at distance at most 1. Thus we can first uncolor all such edges, and then return them their colors one by one, and apply Lemma 6 at every step. So we get an se-coloring of $G$, a contradiction.

In the rest of the proof we will attempt to construct a stable se-coloring of $G_1$ from $L$. For this, fix an se-coloring $\psi$ of $G_2 = G_1 - V(C)$ from $L$ (it exists by the minimality of $G$). Construct the auxiliary graph $H$ with $V(H) = \{v_i v'_i : i = 1, \ldots, t\}$ by making $v_j v'_j$ adjacent in $H$ to $v_i v'_i$ if $j \in \{i - 1, i + 1\}$, or $v'_j = v'_i$ or
By definition, if $H$ has a $L_1$-coloring $\psi'$, then the union $\psi \cup \psi'$ forms a stable se-coloring of $G_1$ contradicting Lemma 8. Thus $H$ has no $L_1$-coloring. But by (3), $L_1$ is a so called degree list for $H$. Since $H$ has Hamiltonian cycle, it is 2-connected.

By a well-known result of Borodin [1] (for a short proof, see [8]), for every 2-connected $H$ and a list $L_1$ satisfying (3), if $H$ has no $L_1$-coloring, then

(i) $|L_1(v_i v'_i)| = d_H(v_i v'_i)$ for every $i = 1, \ldots, t$;

(ii) all lists are the same; and

(iii) $H$ is a complete graph or an odd cycle.

Since $|V(H)| = t$, we have three cases.

**Case 1:** $H = K_t$ for $t \geq 5$. If not all $v'_i$ are distinct, say $v'_1 = v'_2$, then since $C$ is a shortest cycle, $r \leq 3$ and $t - r \leq 1$. Thus then $t \leq 4$, which is not the case. So, all $v'_i$ are adjacent to at most two other vertices $v'_j$. Thus to have $H = K_t$ for $t \geq 5$, we need $t = 5$ and $N_G(v'_i) = \{v_i, v'_{i-2}, v'_{i+2}\}$ for all $i = 1, \ldots, 5$. This means, $G$ is the Petersen graph, and $\psi$ colored the edges of the 5-cycle $C_1 = (v'_1, v'_3, v'_2, v'_4, v'_5)$ so that the lists $L_1(v_i v'_i)$ for all $i = 1, \ldots, 5$ become the same. Since $|L(v'_1 v'_3)| = 8$, we can recolor $v'_1 v'_3$ with another color in $L(v'_4 v'_5)$ distinct from the colors of all edges in $C_1$. Then the list $L_1(v_3 v'_4)$ does not change, but the lists of all other $v_i v'_i$ will change. Thus for the new coloring, condition (ii) will not hold anymore, and we get a stable se-coloring of $G_1$.

**Case 2:** $H = K_4$. If not all $v'_i$ are distinct, say $v'_1 = v'_2$, then since $C$ is a shortest cycle, $r = 3$. But then at most 3 colored edges are incident with $v'_1$ or its neighbor, thus $|L_1(v_1 v'_1)| \geq 5$, a contradiction to (i). So, all $v'_i$ are distinct and $v'_1 v'_3, v'_2 v'_4 \in E(G)$. Since at most 6 colored edges are at distance at most 1 from $v'_1 v'_3$ in $G_2$, we can recolor it with another color from its list distinct from the colors of these at most 6 edges. If after this recoloring, the list $L_1(v_2 v'_3)$ or $L_1(v_2 v'_4)$ does not change, then (ii) does not hold anymore and we can get a stable se-coloring of $G_1$. If both, $L_1(v_2 v'_3)$ and $L_1(v_4 v'_3)$ change, then two edges connect $\{v'_1, v'_3\}$ with $\{v'_2, v'_4\}$. Since $G$ is 3-regular, this means that $G$ has only 8 vertices, and so $|L_1(v_i v'_i)| \geq 4$ for each $i$, contradicting (i).

**Case 3:** $H$ is a cycle with $t$ vertices, where $t$ is odd. Similarly to Case 2, all $v'_i$ are distinct and not adjacent to each other. Also by (ii), we may assume $L_1(v_i v'_i) = \{\alpha, \beta\}$ for all $i = 1, \ldots, t$. We color $v_i v'_i$ with $\alpha$ for $i = 1, 3, 5, \ldots, t$ and with $\beta$ for $i = 2, 4, 6, \ldots, t - 1$. Then we color $v_1 v_2$ with $\gamma_0 \in L(v_1 v_2) - \psi(v'_i) - \psi(v'_i) - \{\alpha, \beta\}$ and $v_1 v_2$ with $\gamma_1 \in L(v_1 v_2) - \{\alpha, \beta, \gamma_0\}$. Now for $i = 2, \ldots, t - 1$, we greedily color $v_i v_{i+1}$ with a color $\gamma_i \in L(v_i v_{i+1}) - \{\alpha, \beta, \gamma_0, \gamma_1, \gamma_{i-2}, \gamma_{i-1}\}$.

Similarly to the end of the proof of Lemma 8, the new coloring is an se-coloring of $G$, since colors $\alpha$ and $\beta$ occur at most three times, and $\psi$ and $\psi'$ are stable stable se-colorings of $G$.
and $\beta$ are not used on the edges distinct from $v_1v'_1, \ldots, v_iv'_i$ at distance at most 1 from any of them. This proves the theorem.

4. Proof of Theorem 4.1

Suppose that the theorem is not true. Let $H$ have the fewest edges among the subcubic graphs with $\text{mad}(H) < \frac{7}{3}$ such that for some list $L$ with $|L(e)| = 5$ for each $e \in E(H)$, $H$ has no se-coloring from $L$. Clearly $H$ is connected.

Claim 9. $H$ has no weak 2-vertices.

**Proof.** Suppose $H$ contains a 2-vertex $u$ adjacent to a 1-vertex $u_1$. Let $u_2$ be the second neighbor of $u$. By the minimality of $H$, graph $H' = H - \{u_1u\}$ has an se-coloring $\phi$ from $L$. We can view $\phi$ as a partial se-coloring of $H$. Since $|\phi(u_2)| \leq 3$, there is $\alpha \in L(u_1u) - \phi(u_2)$. By Lemma 6(a), if we color $u_1u$ with $\alpha$, then we get an se-coloring of $H$ from $L$. ■

Claim 10. $H$ does not contain a 3-vertex adjacent to two 1-vertices.

**Proof.** Suppose that $H$ contains a 3-vertex $u$ with $N(u) = \{u_1, u_2, u_3\}$, where $d(u_1) = d(u_2) = 1$. By the minimality of $H$, graph $H' = H - \{u_1u\}$ has an se-coloring $\phi$ from $L$. As in the proof of Claim 9, we view $\phi$ as a partial se-coloring of $H$. Since $|\phi(u_3)| \leq 3$ and $|\phi(u_2)| = 1$, there is $\alpha \in L(u_1u) - \phi(u_2) - \phi(u_3)$. By Lemma 6(a), if we color $u_1u$ with $\alpha$, then we get an se-coloring of $H$ from $L$. ■

Let $H^*$ denote the graph obtained from $H$ by deleting all vertices of degree 1. By Claims 9 and 10, $\delta(H^*) \geq 2$.

Claim 11. $H^*$ has no 3-cycle $C = xvw$ such that $d_{H^*}(v) = d_{H^*}(w) = 2$.

**Proof.** Suppose that $H$ contains a cycle $xvw$ such that $d_{H^*}(v) = d_{H^*}(w) = 2$. If $z \in \{v, w\}$ has a 1-neighbor in $H - \{v, w\}$, denote this neighbor by $z'$.

If $x$ has a neighbor in $H$ different from $v$ and $w$ we denote it by $t$.

**Case 1:** $H^* = C$. Let $\phi$ be any coloring of the edges of $C$ from the lists such that all three colors are distinct. By definition, this is a partial se-coloring of $H$.

Now consecutively for each $z \in \{x, v, w\}$, color edge $zz'$ (if it exists) with a color in $L(zz') - \{\phi(xv), \phi(vw), \phi(wx)\}$. By Lemma 6(b), at each step we again will obtain a partial se-coloring of $H$. So, after the last step we get an se-coloring of $H$ from $L$, a contradiction.

**Case 2:** The vertex $t$ exists and $d_{H}(t) \geq 2$. Let $H_0 = H - \{v, v', w, w'\}$, note that the vertices $v'$ and $w'$ may not exist.

By the minimality of $H$, graph $H_0$ has an se-coloring $\phi$ from $L$. We view $\phi$ as a partial se-coloring of $H$. Color $vx$ with a color $\alpha_1 \in L(vx) - \phi(t)$ and $wx$ with a
color $\alpha_2 \in L(wx) - \phi(t) - \alpha_1$. By Lemma 6(a), the new partial edge-coloring $\phi'$ is an se-coloring. Now color $vw$ with some $\alpha_3 \in L(vw) - \phi'(t)$. Again by Lemma 6(a), the new partial edge-coloring $\phi''$ is an se-coloring. Then consecutively for $z \in \{v, w\}$, color edge $zz'$ (if it exists) with a color in $L(zz') - \{\alpha_3 \} - \phi(x)$. By Lemma 6(b), at each step we again will obtain a partial se-coloring of $H$. So, after the last step we get an se-coloring of $H$ from $L$, a contradiction.

**Lemma 12.** Graph $H^*$ has no 4-cycle $xuwx$ such that $d_{H^*}(u) = d_{H^*}(v) = d_{H^*}(w) = 2$. Furthermore, if $H^*$ contains a path $xuwy$ such that $d_{H^*}(u) = d_{H^*}(v) = d_{H^*}(w) = 2$, then $d_{H^*}(x) = d_{H^*}(y) = 3$. Moreover, if $N_{H^*}(x) = \{u, x_1, x_2\}$ and $N_{H^*}(y) = \{w, y_1, y_2\}$, then $d_{H^*}(x_1) = d_{H^*}(x_2) = d_{H^*}(y_1) = d_{H^*}(y_2) = 3$.

**Proof.** Suppose that $H$ contains a path $xuwy$ or a cycle $xuwx$ such that $d_{H^*}(u) = d_{H^*}(v) = d_{H^*}(w) = 2$. If $u$ has a 1-neighbor in $H$, we will denote this neighbor by $u'$. The vertices $v'$ and $w'$ are defined similarly.

Now we will prove that the vertex $v'$ does not exist. Otherwise, consider $H' = H - v'$. By the minimality of $H$, graph $H'$ has an se-coloring $\phi$ from $L$. We view $\phi$ as a partial se-coloring of $H$. By Lemma 6(b), the coloring $\phi'$ obtained from $\phi$ by coloring $vv'$ with a color in $L(vv') - \{\phi(xu), \phi(uv), \phi(vw), \phi(wx)\}$ if we have a path (or a color in $L(vv') - \{\phi(xu), \phi(uv), \phi(vw), \phi(wx)\}$ if we have a 4-cycle) is a se-coloring from $L$ of the whole $H$. This contradicts the choice of $H$. So

$$d_{H}(v) = 2. \quad (4)$$

**Case 1:** $H^*$ contains a cycle $C = xuvw$ such that $d_{H^*}(u) = d_{H^*}(v) = d_{H^*}(w) = 2$. Let $t$ be the third neighbor of $x$ in $H$, if it exists.

**Case 1.1:** $H^* = C$. Let $\phi$ be any coloring of the edges of $C$ from the lists such that all four colors are distinct. By definition, this is a partial se-coloring of $H$. Now consecutively for each $z \in \{u, w\}$, color the edge $zz'$ (if it exists) with a color in $L(zz') - \{\phi(xu), \phi(uv), \phi(vw), \phi(wx)\}$. If $xt$ exits color the edge $xt$ with a color $L(xt) - \{\phi(xu), \phi(uv), \phi(vw), \phi(wx)\}$ By Lemma 6(b), at each step we again will obtain a partial se-coloring of $H$. So, after the last step we get an se-coloring of $H$ from $L$, a contradiction.

**Case 1.2:** The vertex $t$ exists and $d_{H}(t) \geq 2$. Let $H_0 = H - \{u, v, w, v', w'\}$. By the minimality of $H$, graph $H_0$ has an se-coloring $\phi$ from $L$. We view $\phi$ as a partial se-coloring of $H$. Color $uw$ with a color $\alpha_1 \in L(uw) - \phi(t) - \alpha_1$. By Lemma 6(a), the new partial edge-coloring $\phi'$ is an se-coloring. Now color $vw$ with some $\alpha_3 \in L(vw) - \phi'(x)$ and $uv$ with some $\alpha_4 \in L(uv) - \phi'(x) - \alpha_3$. Again by Lemma 6(a), the new partial edge-coloring $\phi''$ is an se-coloring. Then consecutively for $z \in \{u, w\}$, color edge $zz'$ (if it exists) with a color in $L(zz') - \{\alpha_1, \alpha_2, \alpha_3, \alpha_4\}$. By Lemma 6(b), at each step we again
will obtain a partial se-coloring of $H$. So, after the last step we get an se-coloring of $H$ from $L$, a contradiction.

**Case 2:** $H^*$ contains a path $P = xuwy$ such that $d_{H^*}(u) = d_{H^*}(v) = d_{H^*}(w) = 2$. Let $N_H(y) \subseteq \{w, y_1, y_2\}$ (maybe only one of $y_1$, $y_2$ exists) and $N_H(x) \subseteq \{u, x_1, x_2\}$. Let $H_1 = H - \{v, u', w'\}$. By the minimality of $H$, graph $H_1$ has an se-coloring $\phi$ from $L$. We view $\phi$ as a partial se-coloring of $H$. Let $A(wv) = L(wv) - \phi(y)$, $A(wu') = L(wu') - \phi(y)$, $A(uv) = L(uv) - \phi(x)$ and $A(uu') = L(uu') - \phi(x)$. Since $|\phi(z)| \leq 3$ for every $z \in V(H)$, each of $A(wv), A(wu'), A(uv)$ and $A(uu')$ contains at least two colors. (5)

**Case 2.1:** Suppose $|A(wv) \cup A(wu')| + |A(uv) \cup A(uu')| \geq 5$. By (5) and symmetry, we may assume $|A(uv) \cup A(uu')| \geq 3$. Color $uv$ with a color $\alpha_1 \in A(uv) - \phi(xu)$ and $wu'$ with a color $\alpha_2 \in A(wu') - \alpha_1$. Since edge $uw$ is not colored, by Lemma 6(a), the new partial edge-coloring $\phi_1$ is an se-coloring. By (5) and the fact that $|A(uv) \cup A(uu')| \geq 3$, we can choose distinct $\alpha_3 \in A(uv) - \alpha_1$ and $\alpha_4 \in A(uu') - \alpha_1$. Let $\phi_2$ be obtained from $\phi_1$ by coloring $uw$ with $\alpha_3$. We claim that $\phi_2$ is a partial se-coloring of $H$. (6)

Indeed, suppose there is a color $\beta$ and either a path $z_1 z_2 z_3 z_4 z_5$ or a cycle $z_1 z_2 z_3 z_4 z_1$ containing edge $uv$ whose edges are colored with $\alpha_3$ and $\beta$. By symmetry, we may assume that $\{u, v\} = \{z_i, z_{i+1}\}$ for some $i \in \{1, 2\}$. Then $\phi(z_{i+2} z_{i+3}) = \alpha_3$. Since $\alpha_3 \in A(uv) = L(uv) - \phi(x)$, this yields $z_{i+2} = w$ and thus $u = z_i, v = z_{i+1}$. Since $\phi_1(wv) = \alpha_1 \neq \phi_1(xu), \beta = \alpha_1, i = 1$ and we have no bicolored cycles. Since $i = 1, z_i = w$. So $z_4 = y$ and $z_5 \in \{y_1, y_2\}$. But $\alpha_1 \notin \phi(y)$. This contradiction proves (6).

Now, let $\phi_3$ be obtained from $\phi_2$ by coloring $uu'$ with $\alpha_4$. By (6) and Lemma 6(b), $\phi_3$ is a partial se-coloring of $H$. But by (4), $\phi_3$ colors all edges of $H$. This contradiction proves Case 2.1.

If Case 2.1 does not hold, then by (5), we may assume that $A(uv) = A(uu') = \{\alpha_1, \alpha_2\}$ and $A(wv) = A(wu') = \{\beta_1, \beta_2\}$. This means that $L(uv) = L(uu') = \{\alpha_1, \alpha_2\} \cup \phi(x)$ and $L(wv) = L(wu') = \{\beta_1, \beta_2\} \cup \phi(y)$. (7)

In particular, $d_H(x) = d_H(y) = 3$.

**Case 2.2:** $\{\alpha_1, \alpha_2\} \cap \{\beta_1, \beta_2\} = \emptyset$. By symmetry, we may assume that $\alpha_1 \neq \phi(wy)$ and $\beta_1 \neq \phi(xu)$. Let $\phi_1$ be obtained from $\phi$ by coloring $uw$ with $\alpha_1$ and $vw$ with $\beta_1$. By Lemma 6(a), $\phi_1$ is a partial se-coloring of $H$. Then let $\phi_2$ be obtained from $\phi_1$ by coloring $uu'$ with $\alpha_2$ and $wu'$ with $\beta_2$. Again by Lemma 6(a), $\phi_2$ is a partial se-coloring of $H$. By (4), $\phi_2$ colors all edges of $H$, contradicting the choice of $H$.
Case 2.3: $\{\{\alpha_1, \alpha_2\} \cap \{\beta_1, \beta_2\}\} = 1$. By (7), we may assume that $L(ww') = L(ww') = \{1, 2, 3, 4, 5\}$, $\alpha_1 = \beta_1 = 1$, $\beta_2 = 2$, $\phi(\alpha y) = 3$, $\phi(\beta y) = 4$ and $\phi(\gamma y) = 5$. By the case, $\alpha_2 \neq 2$. Let $\phi_1$ be obtained from $\phi$ by setting $\phi_1(ww) = 2$ and $\phi_1(ww) = 1$ (in this order). Then we get partial se-colorings after both steps by Lemma 6(a), since $1 \notin \phi(y) \cup \phi(x)$. Let $\phi_2$ be obtained from $\phi_1$ by setting $\phi_2(ww') = \alpha_2$. If $\phi_2$ has a bicolored path $z_1z_2z_3z_4z_5$ with $z_1z_2 = u'u$, then the second edge should be $ww$, since $\alpha_2 \notin \phi(x)$. But then the third edge must be $ww$ and $\phi_1(ww) = 2$ and $\alpha_2 \neq 2$. Hence no such a bicolored path exists. Thus $\phi_2$ is a partial se-coloring of $H$. So if $3 \notin \phi(y_1)$, then by coloring $ww'$ with 4, we obtain from $\phi_2$ an se-coloring of $H$, a contradiction. Thus $3 \notin \phi(y_1)$. Similarly, $3 \in \phi(y_2)$.

Let $\gamma_1, \gamma_2 \in L(ww) = \{3, 4, 5\}$. Return to coloring $\phi$. Suppose $\gamma_1 \notin \phi(y_1) \cup \phi(y_2)$. We recolor $ww$ with $\gamma_1$, color $ww$ with $\gamma_2$, $uv$ with a color $\alpha \in \{1, \alpha_2\} - \gamma_1$, and $uu'$ with $\alpha' \in \{1, \alpha_2\} - \alpha$ (in this order). After each step, by Lemma 6(a), we get a partial se-coloring of $H$. So the resulting coloring $\phi_3$ is a partial se-coloring of $H$ in which only $ww'$ is uncolored. Now after coloring $ww'$ with $\lambda \in \{4, 5\} - \phi_3(ww)$ we get an se-coloring of $H$ from $L$, a contradiction. Thus by the symmetry between $\gamma_1$ and $\gamma_2$, $\{\gamma_1, \gamma_2\} \subset \phi(y_1) \cup \phi(y_2)$. In particular, this means $d_H(y_1) = d_H(y_2) = 3$. Let $N_H(y_1) = \{y, y_3, y_4\}$ and $N_H(y_2) = \{y, y_5, y_6\}$. We may assume that $\phi(y_1y_3) = \phi(y_2y_5) = 3$, $\phi(y_1y_4) = \gamma_1$ and $\phi(y_2y_6) = \gamma_2$.

If $4 \notin \phi(y_4)$, consider the se-coloring $\phi_4$ from the previous paragraph, but now color $ww'$ with 5. Since $\gamma_1 \notin \phi(y_2)$ and $2 \notin \{\alpha_1, \alpha_2\}$, the only possible bicolored path with 4 edges is $w'wwux$. This means $\phi(xu) = 2$ and $\alpha_2 = \phi_3(ww) = 5$. In this case, recolor $ww$ with 3. Thus $4 \in \phi(y_4)$, and in particular, $d_H(y_4) \geq 2$, so $y_4 \in V(H^*)$. Similarly, $5 \in \phi(y_5)$, and so $y_5 \in V(H^*)$. We claim that also

$$\{y_3, y_5\} \subset V(H^*).$$

Suppose (8) fails, say $d_H(y_5) = 1$. Consider again the partial se-coloring $\phi_2$.

Recolor $y_5y_2$ with a $\lambda \in L(ww) = \{3, 5\} - \phi(y_6)$ (since $5 \in \phi(y_6)$, this set is nonempty) and color $ww'$ with 5. If there is a bicolored 4-path $z_1z_2z_3z_4z_5$ with $z_1 = y_5$ and $z_2 = y_2$, then since $\lambda \notin \phi(y_6)$, $z_3 = y$. Since $\lambda \neq 3$, $z_4 = y_1$ and $\lambda = 4$. But $5 \notin \phi(y_1)$ since $\gamma_1 \notin \{3, 4, 5\}$. So we obtain an se-coloring of $H$ from $L$, contradicting the choice of $H$. This proves (8). This together with $y_4, y_6 \in V(H^*)$ shows $d_H(y) = d_H(y_1) = d_H(y_2) = 3$. By symmetry also $d_H(x) = d_H(x_1) = d_H(x_2) = 3$, and so the lemma holds in this case.

Case 2.4: $\{\alpha_1, \alpha_2\} = \{\beta_1, \beta_2\}$. By (7), we may assume that $L(ww) = L(ww') = \{1, 2, 3, 4, 5\}$, $\alpha_1 = \beta_1 = 1$, $\alpha_2 = \beta_2 = 2$, $\phi(\gamma y) = 3$, $\phi(\alpha y) = 4$ and $\phi(\gamma y) = 5$. Consider the partial se-coloring $\phi_4$ defined in Case 2.3. Let $\phi_4$ be obtained from $\phi_4$ by coloring $uu'$ with 2. If there is a bicolored 4-path $z_1z_2z_3z_4z_5$ with $z_1 = u'$ and $z_2 = u$, then since $2 \notin \phi(x)$, $z_3 = v$ and so $z_4 = w$. But $\phi(ww) = 3 \neq 1$. Thus $\phi_4$ is a partial se-coloring of $H$. Repeating the argument
of the end of the first paragraph of Case 2.3, we conclude that $3 \in \phi(y_1)$ and $3 \in \phi(y_2)$.

Let $\gamma_1, \gamma_2 \in L(wy) - \{3, 4, 5\}$. Return to coloring $\phi$. Suppose $\gamma_1 \notin \phi(y_1) \cup \phi(y_2)$. We uncolor $wy$, color $vw$ with $\lambda \in \{4, 5\} - \phi(xu)$, $uw'$ with $\lambda' \in \{4, 5\} - \lambda$, $uv$ with a color $\alpha \in \{1, 2\} - \gamma_1$, $uw'$ with $\alpha' \in \{1, 2\} - \alpha$ and finally $wy$ with $\gamma_1$ (in this order). After each step, by Lemma 6(a), we get a partial se-coloring of $H$. So the resulting coloring $\phi_3$ is an se-coloring of $H$, a contradiction. Thus by the symmetry between $\gamma_1$ and $\gamma_2$, $\{\gamma_1, \gamma_2\} \subseteq \phi(y_1) \cup \phi(y_2)$. In particular, this means $d_H(y_1) = d_H(y_2) = 3$. Let $N_H(y_1) = \{y, y_3, y_4\}$ and $N_H(y_2) = \{y, y_5, y_6\}$.

We may assume that $\phi(y_1y_3) = \phi(y_2y_5) = 3$, $\phi(y_1y_4) = \gamma_1$ and $\phi(y_2y_6) = \gamma_2$. If $4 \notin \phi(y_1)$, consider the se-coloring $\phi_3$ from the previous paragraph, in which we recolor the edge $e \in \{vw, uw'\}$ of color 4 with 3. We will get an se-coloring of $H$ from $L$, unless $e = vw$ and $\phi(xu) = 3$. But in this case, we recolor $vw$ with 5 and $uw'$ with 3 (i.e., switch the colors of $vw$ and $uw'$). Thus $4 \in \phi(y_4)$. Similarly, $5 \in \phi(y_6)$. As in Case 2.3, we claim that also (8) holds and the proof word by word repeats such proof in Case 2.3. So we again get $d_{H^*}(y) = d_{H^*}(y_1) = d_{H^*}(y_2) = 3$ and by symmetry $d_{H^*}(x) = d_{H^*}(x_1) = d_{H^*}(x_2) = 3$. This proves the lemma. 

**Lemma 13.** $H^*$ does not contain a 3-vertex adjacent to three 2-vertices such that at least two of these vertices have 2-neighbors in $H^*$. 

**Proof.** Suppose that $H^*$ contains a 3-vertex $u$ adjacent to 2-vertices $x, y, z$ such that $y$ has a 2-neighbor $y_1$ and $z$ has a 2-neighbor $z_1$. By Claim 11, $y_1, z_1 \notin \{x, y, z\}$. By Lemma 12, $y_1 \neq z_1$. Let $w$ (respectively $t$) denote the second neighbor in $H^*$ of $y_1$ (respectively, $z_1$). For each $r \in \{x, y, y_1, z, z_1\}$, if $r$ has a (unique) 1-neighbor in $H$, then we denote this neighbor by $r'$ (see Figure 2). Let $v$ be the neighbor of $x$ different from $x'$ and $u$.

Let $H_1 = H - \{u, x', y, y', z, z', y_1, z_1\}$. By the minimality of $H$, graph $H_1$ has an se-coloring $\phi$ from $L$. We view $\phi$ as a partial se-coloring of $H$. Let $A(xu) = L(xu) - \phi(v), A(xx') = L(xx') - \phi(v), A(yy_1) = L(yy_1) - \phi(w), A(yy_1') = L(yy_1') - \phi(w), A(zz_1) = L(zz_1) - \phi(t)$ and $A(z_1z_1') = L(z_1z_1') - \phi(t)$. Similarly to (5), we have

$$
\text{each of } A(xu), A(xx'), A(yy_1), A(yy_1'), A(zz_1) \\
\text{and } A(z_1z_1') \text{ contains at least two colors. (9)}
$$

**Case 1:** There is $\alpha \in A(yy_1) \cap A(zz_1)$. Color $yy_1$ and $zz_1$ with $\alpha$, then color $xu$ with a color $\beta \in A(xu) - \alpha$, then $y_1y_1'$ with $\alpha_1 \in A(y_1y_1') - \alpha$, $z_1z_1'$ with $\alpha_2 \in A(z_1z_1') - \alpha$ and $xx'$ with $\beta' \in A(xx') - \beta$. Since edges $uz$ and $uy$ are not colored, by Lemma 6(a), the new partial edge-coloring $\phi_1$ of $H$ is an se-coloring. Then we color $uy$ with $\gamma_1 \in L(uy) - \{\alpha, \beta, \phi(xv)\}$ and $uz$ with $\gamma_2 \in L(uz) - \{\alpha, \beta, \phi(xv), \gamma_1\}$. Let $\phi_2$ be the new coloring. If Lemma 6(b) does
not apply to \( \phi_2(zu) \), then \( \phi_2(zu) = \phi_1(tz_1) \). But the color \( \alpha \) of \( z_1z \) is not in \( \phi_2(u) \cup \phi_2(t) \) by definition. So there is no bicolored 4-path in \( \phi_2 \) containing \( uz \).

Similarly, there is no bicolored 4-path in \( \phi_2 \) containing \( uy \). Thus, \( \phi_2 \) is a partial se-coloring of \( H \). Finally, color \( yy' \) with a \( \lambda_1 \in L(yy') - \{ \alpha, \beta, \phi_2(uy), \phi_2(uz) \} \) and \( zz' \) with a \( \lambda_2 \in L(zz') - \{ \alpha, \beta, \phi_2(uy), \phi_2(uz) \} \). Let \( \phi_3 \) be the new coloring. As above, if Lemma 6(b) does not apply to \( \phi_3(zz') \), then \( \phi_3(zz') = \phi_1(tz_1) \). But the color \( \alpha \) of \( z_1z \) is not in \( \phi_3(t) \) by definition. So there is no bicolored 4-path in \( \phi_3 \) containing \( zz' \). Similarly, there is no bicolored 4-path in \( \phi_3 \) containing \( yy' \).

Thus, \( \phi_3 \) is an un-coloring of \( H \), a contradiction.

\[ \]

\textbf{Case 2:} \( A(yy_1) \cap A(zz_1) = \emptyset \). Color \( xu \) with a color \( \beta \in A(xu) \), then color \( yy_1 \) with a color \( \alpha_1 \in A(yy_1) - \beta \), then \( zz_1 \) with a color \( \alpha_2 \in A(zz_1) - \beta \), then \( y_1y_1' \) with \( \alpha_1' \in A(y_1y_1') - \alpha_1 \), \( z_1z_1' \) with \( \alpha_2' \in A(z_1z_1') - \alpha_2 \) and \( xx' \) with \( \beta' \in A(xx') - \beta \).

Similarly to Case 1, by Lemma 6(a), the new partial edge coloring \( \phi_1 \) of \( H \) is an se-coloring. Then we color \( uy \) with \( \gamma_1 \in L(uy) - \{ \alpha_1, \alpha_2, \beta, \phi(xu) \} \) and \( uz \) with \( \gamma_2 \in L(uz) - \{ \alpha_1, \alpha_2, \beta, \gamma_1 \} \). Let \( \phi_2 \) be the new coloring. If Lemma 6(b) does not apply to \( \phi_2(zu) \), then \( \phi_2(zu) \in \{ \phi_1(tz_1), \phi_1(xu) \} \). But the color \( \alpha_2 \) of \( z_1z \) is not in \( \phi_2(u) \cup \phi_2(t) \), and the color \( \beta \) of \( xu \) is not in \( \phi_2(v) \cup \phi_2(z) \), by definition. So there is no bicolored 4-path in \( \phi_2 \) containing \( uz \). Similarly, there is no bicolored 4-path in \( \phi_2 \) containing \( uy \). Thus, \( \phi_2 \) is a partial se-coloring of \( H \). Finally, color \( yy' \) with a \( \lambda_1 \in L(yy') - \{ \alpha_1, \beta, \phi_2(uy), \phi_2(uz) \} \) and \( zz' \) with a \( \lambda_2 \in L(zz') - \{ \alpha_2, \beta, \phi_2(uy), \phi_2(uz) \} \). Let \( \phi_3 \) be the new coloring. As above, if Lemma 6(b) does not apply to \( \phi_3(zz') \), then \( \phi_3(zz') = \phi_1(tz_1) \). But the color \( \alpha_2 \) of \( z_1z \) is not in \( \phi_2(t) \) by definition. So there is no bicolored 4-path in \( \phi_3 \) containing \( zz' \). Similarly, there is no bicolored 4-path in \( \phi_3 \) containing \( yy' \). Thus, \( \phi_3 \) is an se-coloring of \( H \), a contradiction.

\[ \]

We will now show that \( |E(H^*)| \geq \frac{2}{3}|V(H^*)| \), which will contradict the fact that \( \text{mad}(H) < \frac{2}{3} \). For this, we will use the discharging method. First, recall that

\[ \]
by Claim 9 and Claim 10, \( \delta(H^*) \geq 2 \). Also, by Lemma 12, for each path \( uvw \) in \( H^* \),

if \( d_{H^*}(u) = d_{H^*}(v) = d_{H^*}(w) = 2 \), then \( u \) and \( w \) have distinct 3-neighbors in \( H^* \).

(10)

For each vertex \( v \) of \( H^* \), we define the charge of \( v \) as \( \omega(v) = d_H(v) - \frac{7}{3} \). So

\[
\sum_{v \in V(H^*)} \omega(v) = \sum_{v \in V(H^*)} d_{H^*}(v) - \frac{7}{3} |V(H^*)| = 2|E(H^*)| - \frac{7}{3} |V(H^*)|. \tag{11}
\]

During the discharging process, we will modify \( \omega \) to a new charge \( \omega^* \) so that the total sum of charges will not change. On the other hand, we will show that \( \omega^*(v) \geq 0 \) for all \( v \in V(H^*) \). By (11), this will yield \( |E(H^*)| \geq \frac{7}{6} |V(H^*)| \) contradicting \( \text{mad}(H) < \frac{7}{3} \).

The discharging rules are as follows:

(R1) Every 2-vertex in \( H^* \) adjacent to two 3-vertices receives \( \frac{1}{6} \) from each of the two neighbors.

(R2) Every 2-vertex in \( H^* \) adjacent to exactly one 3-vertex receives \( \frac{1}{3} \) from this 3-vertex.

(R3) Every 2-vertex in \( H^* \) adjacent to two 2-vertices, say \( x \) and \( y \) receives \( \frac{1}{6} \) from the other neighbor of \( x \) and \( \frac{1}{6} \) from the other neighbor of \( y \). Note that by (10), these "other neighbors" are distinct 3-vertices in \( H^* \).

By (R1)--(R3) none of the 2-vertices in \( H^* \) gives away any charge, and each of them receives charge exactly \( \frac{1}{3} \) from other vertices. Thus \( \omega^*(v) = 0 \) for each 2-vertex \( v \).

Now, let \( v \) be a 3-vertex in \( H^* \). If \( v \) has no 2-neighbors, then it keeps its charge \( \frac{2}{3} \). If \( v \) has exactly one 2-neighbor, then by (R1)--(R3), it gives away at most \( \frac{1}{3} + \frac{1}{6} \) and is left with charge at least \( \frac{2}{3} - \frac{1}{3} - \frac{1}{6} = \frac{1}{3} \). If \( v \) has exactly two 2-neighbors, then by Lemma 12, Rule (R3) does not apply to \( v \). Thus in this case \( v \) gives away at most \( \frac{1}{3} + \frac{1}{3} \) and is left with charge at least 0. Finally, suppose \( v \) has three 2-neighbors. Again by Lemma 12, Rule (R3) does not apply to \( v \). Moreover, by Lemma 13, at most one 2-neighbor of \( v \) has also a 2-neighbor. This means that (R2) applies to \( v \) at most once. So, \( v \) is left with charge at least \( \frac{2}{3} - \frac{1}{3} - \frac{1}{6} - \frac{1}{6} = 0 \). This completes the proof of Theorem 4.1.
5. Proof of Theorem 4.2

Suppose that the theorem is not true. Let $H$ have the fewest edges among the
subcubic graphs with $\text{mad}(H) < \frac{3}{2}$ such that for some list $L$ with $|L(e)| = 6$ for
each $e \in E(H)$, $H$ has no se-coloring from $L$. Clearly $H$ is connected.

The Claim 9, and 10 hold for the graph $H$ since they hold for such graph no
matter what is the mad. Then we have:

Claim 14. $H$ has no weak 2-vertices.

Claim 15. $H$ does not contain a 3-vertex adjacent to two 1-vertices.

So, as in the previous section, the graph $H^*$ obtained from $H$ by deleting all
vertices of degree 1, has minimum degree at least two.

Lemma 16. $H^*$ does not contain a 2-vertex adjacent to a 2-vertex.

Proof. Suppose that $H$ contains a path $xuvy$ or a cycle $xuvx$ such that $d_{H^*}(u) =
d_{H^*}(v) = 2$. If $u$ (respectively, $v$) has a 1-neighbor in $H$, denote this neighbor by
$u'$ (respectively, by $v'$), otherwise it does not exist.

Case 1: $H^*$ contains a cycle $C = xuvx$ such that $d_{H^*}(u) = d_{H^*}(v) = 2$. Let
$w$ be the third neighbor of $z$ in $H$, if it exists. If $H^* = C$, then $H$ has at most
6 edges, and we can greedily color them with all colors distinct. So, $H^* \neq C$,
and thus the vertex $w$ exists and $d_H(w) \geq 2$. Let $H_0 = H - \{u, u', v, v'\}$. By
the minimality of $H$, graph $H_0$ has an se-coloring $\phi$ from $L$. We view $\phi$ as a
partial se-coloring of $H$. Color $ux$ with a color $\alpha_1 \in L(ux) - \phi(u)$, then $uv$
with a color $\alpha_2 \in L(vx) - \phi(v) - \alpha_1$, and then color $uv$ with a color $\alpha_3 \in
L(vy) - \phi(v) - \alpha_1 - \alpha_2$. By Lemma 6(a), the new partial edge-coloring $\phi'$ of $H$
is an se-coloring. Now consecutively for $z \in \{u, v\}$, color edge $zz'$ (if it exists) with
a color in $L(zz') - \phi'(x) - \alpha_3$. By Lemma 6(b), at each step we again will obtain
a partial se-coloring of $H$. So, after the last step we get an se-coloring of $H$ from
$L$, a contradiction.

Case 2: $H^*$ contains a path $P = xuvy$ such that $d_{H^*}(u) = d_{H^*}(v) = 2$. Let
$N_H(y) \subseteq \{v, y_1, y_2\}$ (maybe only one of $y_1, y_2$ exists) and $N_H(x) \subseteq \{u, x_1, x_2\}$. Let $H_1 = H - \{u', v'\} - uv$. Similarly to Case 1, $H_1$ has an se-coloring $\psi$ from $L$. We view $\psi$ as a partial se-coloring of $H$. First, we try to extend $\psi$ to $uv$. If there
is $\alpha_1 \in L(uv) - \psi(x) - \psi(y)$, then we color $uv$, which by Lemma 6(a), would yield
a new partial se-coloring of $H$. Otherwise, $L(uv) \subseteq \psi(x) \cup \psi(y)$, which yields
that $\psi(x)$ and $\psi(y)$ are disjoint sets of size 3 each. So, we may assume that

\[ L(uv) = \{1, \ldots, 6\}, \text{ where } \psi(xu) = 1, \psi(xx_1) = 2, \]
\[ \psi(xx_2) = 3, \psi(yy_1) = 4, \psi(yy_2) = 5, \psi(vy) = 6. \]

(12)

In particular, $d_H(x) = d_H(y) = 3$. For $i = \{1, 2\}$, let $N_H(y_i) = \{y, z_i, t_i\}$ (see
Figure3). If coloring $uv$ with 4 does not create a bicolored 4-path, we do this.
Otherwise, this is a path of colors 4 and 6, and so $6 \in \psi(y_1)$. Similarly, after trying to color $uv$ with 5, we conclude that $6 \in \psi(y_1)$ and so $|\psi(y_1) \cup \psi(y_2)| \leq 5$.

Figure 3. Two adjacent 2-vertices in $H^*$.

So we may recolor $vy$ with $\alpha_2 \in L(vy) - (\psi(y_1) \cup \psi(y_2))$ and color $uv$ with 6. By the definition of $\alpha_2$ and the fact that all colors 1, …, 6 are distinct, the new edge-coloring $\psi'$ is a partial se-coloring of $H$ from $L$.

Now we simply color $uu'$ (if exists) with $\alpha_3 \in L(uu') - \psi'(x) - \psi'(v)$ and $vv'$ (if exists) with $\alpha_4 \in L(vv') - \psi'(u) - \psi'(y)$ (note that we allow $\alpha_4 = \alpha_3$). By Lemma 6(b), this yields an se-coloring of $H$ from $L$, a contradiction.

**Lemma 17.** $H^*$ does not contain a 3-vertex adjacent to three 2-vertices.

**Proof.** Suppose that $H^*$ contains a 3-vertex $v$ adjacent to three 2-vertices $x_1$, $x_2$ and $x_3$ whose second neighbors in $H^*$ are $y_1$, $y_2$ and $y_3$, respectively. By Lemma 16, $d_{H^*}(y_i) = 3$ for all $i = 1, 2, 3$. So for $i = 1, 2, 3$, let $N_H(y_i) = \{x_i, u_i, w_i\}$ (some of these vertices $y_i$ may coincide). Also, for $i = 1, 2, 3$, let $x'_i$ denote the neighbor of degree 1 of $x_i$ in $H$, if exists (see Figure 4).

Figure 4. Forbidden configuration of Lemma 17 in $H$.

By the minimality of $H$, graph $H_0 = H - \{v, x'_1, x'_2, x'_3\}$ has an se-coloring $\phi$ from $L$. We view $\phi$ as a partial se-coloring of $H$. If for some $i \in \{1, 2, 3\}$, color $\phi(x_iy_i)$ is present in both, $\phi(u_i)$ and $\phi(w_i)$, then we can recolor $x_iy_i$ with a color...
in \( L(x_iy_i) - (\phi(u_i) \cup \phi(w_i)) \). Thus by the symmetry between \( u_i \) and \( w_i \), we may assume that
\[
\phi(x_iy_i) \notin \phi(u_i) \quad \text{for all } i \in \{1, 2, 3\}.
\] (13)
We will extend \( \phi \) to the whole \( H \) in two steps.

**Step 1:** We extend \( \phi \) to the edges incident with \( v \). We color \( vx_1 \) with \( \beta_1 \in L(vx_1) - \phi(y_1) - \phi(y_2x_2) - \phi(y_3x_3), \) then color \( vx_2 \) with \( \beta_2 \in L(vx_2) - \phi(y_2) - \phi(y_3x_3) - \beta_1, \) and then \( vx_3 \) with \( \beta_3 \in L(vx_3) - \phi(y_3) - \beta_1 - \beta_2. \) We claim that the resulting coloring \( \phi' \) is a partial se-coloring of \( H \). Indeed, if not, then for some \( i \in \{1, 2, 3\} \), edge \( vx_i \) is in a bicolored path or cycle \( P \) with 4 edges. Since \( \beta_i \notin \phi(y_i) \), the second edge of the color \( \beta_i \) in \( P \) must be \( x_jy_j \) for some \( j \neq i \).

Then edge \( vx_j \) also in \( P \). By the symmetry between \( i \) and \( j \), we conclude that \( x_iy_i \) is in \( P \) and may assume \( i < j \). But then by the definition of \( \beta_i \), it differs from \( \phi(x_jy_j) \), a contradiction.

**Step 2:** We extend \( \phi' \) to those of \( x_ix'_i \) that exist. For each such \( i \), we color \( x_ix'_i \) with a color \( \gamma_i \in L(x_ix'_i) - \phi'(v) - \{\phi'(x_iy_i), \phi'(y_iw_i)\}. \) If the resulting coloring \( \phi'' \) is not an se-coloring of \( H \), then for some \( i \in \{1, 2, 3\} \) there is a bicolored 4-path \( P \) starting from \( x'_i \). Since \( \gamma_i \notin \phi'(v) \), the second edge of color \( \gamma_i \) in \( P \) is incident with \( y_i \). Since \( \gamma_i \) was chosen distinct from \( \phi'(y_iw_i) \), this second edge is \( y_iw_i \). But this contradicts (13).

For \( j \in \{1, 2, 3\} \), let \( V_j \) denote the set of vertices of degree \( j \) in \( H^* \). As it was mentioned above, by Claims 14 and 15, \( V_1 = \emptyset \). By Lemma 16, every \( v \in V_2 \) has two neighbors in \( V_3 \), and by Lemma 17, every \( v \in V_3 \) has at most two neighbors in \( V_2 \). It follows that \( |V_3| \geq |V_2| \), which yields \( \text{mad}(H^*) \geq 5/2 \). This proves Theorem 4.2.

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**References**


