Packing chromatic number of cubic graphs

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1. Introduction

For a positive integer \( i \), a set \( S \) of vertices in a graph \( G \) is \( i \)-independent if the distance in \( G \) between any two distinct vertices of \( S \) is at least \( i + 1 \). In particular, a 1-independent set is simply an independent set.

A packing \( k \)-coloring of a graph \( G \) is a partition of \( V(G) \) into sets \( V_1, \ldots, V_k \) such that for each \( 1 \leq i \leq k \) the distance between any two distinct \( x, y \in V_i \) is at least \( i + 1 \). The packing chromatic number, \( \chi_p(G) \), of a graph \( G \) is the minimum \( k \) such that \( G \) has a packing \( k \)-coloring.

The notion of packing \( k \)-coloring was introduced in 2008 by Goddard, Hedetniemi, Hedetniemi, Harris and Rall [15] (under the name broadcast coloring) motivated by frequency assignment problems in broadcast networks. The concept has attracted a considerable attention recently: there are more than 25 papers on the topic (see e.g. [1,5–12,14,21] and references in them). In particular, Fiala and Golovach [10] proved that finding the packing chromatic number of a graph is NP-hard even in the class of trees. Sloper [21] showed that there are graphs with maximum degree 4 and arbitrarily large packing chromatic number.

The question whether the packing chromatic number of all subcubic graphs (i.e., the graphs with maximum degree at most 3) is bounded by a constant was not resolved. For example, Brešar, Klavžar, Rall, and Wash [7] wrote: ‘One of the intriguing problems related to the packing chromatic number is whether it is bounded by a constant in the class of all cubic graphs’. It was proved in [7,17–19,21] that it is indeed bounded in some subclasses of subcubic graphs. On the other hand, Gastineau and Togni [14] constructed a cubic graph \( G \) with \( \chi_p(G) = 13 \), and asked whether there are cubic graphs with a larger packing chromatic number. Brešar, Klavžar, Rall, and Wash [8] answered this question in affirmative by constructing a cubic graph \( G' \) with \( \chi_p(G') = 14 \). The main result of this paper answers the question in full: Indeed, there are cubic graphs with arbitrarily large packing chromatic number. Moreover, we prove that ‘many’ cubic graphs have ‘high’ packing chromatic number:

**Theorem 1.** For each fixed integer \( k \geq 12 \) and \( g \geq 2k + 2 \), almost every \( n \)-vertex cubic graph \( G \) of girth at least \( g \) satisfies \( \chi_p(G) > k \).
The theorem will be proved in the language of the so-called Configuration model, $\mathcal{F}_3(n)$. We will discuss this concept and some important facts on it in the next section. In Section 3 we give upper bounds on the sizes $c_i$ of maximum $i$-independent sets in almost all cubic $n$-vertex graphs of large girth. The original plan was to show that for a fixed $k$ and large $n$, the sum $c_1 + \cdots + c_k$ is less than $n$. But we were not able to prove it (and maybe this is not true). In Section 4, we give an upper bound on the size of the union of an 1-independent, a 2-independent, and a 4-independent sets which is less than $c_1 + c_2 + c_4$. This allows us to prove Theorem 1 in the last section.

2. Preliminaries

2.1. Notation

We mostly use standard notation. If $G$ is a (multi)graph and $v, u \in V(G)$, then $E_G(v, u)$ denotes the set of all edges in $G$ connecting $v$ and $u$, $e_G(v, u) := |E_G(v, u)|$, and $deg_G(v) := \sum_{u \in V(G)} e_G(v, u)$. For $A \subseteq V(G)$, $G[A]$ denotes the sub(multi)graph of $G$ induced by $A$. The independence number of $G$ is denoted by $\alpha(G)$. For $k \in \mathbb{Z}_{>0}$, $[k]$ denotes the set $\{1, \ldots, k\}$.

2.2. The configuration model

The configuration model is due in different versions to Bender and Canfield [2] and Bollobás [3,4]. Our work is based on the version of Bollobás. Let $V$ be the vertex set of the graph, we are going to associate a 3-element set to each vertex in $V$. Let $n$ be an even positive integer. Let $V_n = [n]$ and consider the Cartesian product $W_n = V_n \times V_n$. A configuration/pairing (of order $n$ and degree 3) is a partition of $W_n$ into $3n/2$ pairs, i.e., a perfect matching of elements in $W_n$. There are

$$\left(\frac{3n}{2}\right) \cdot \left(\frac{3n-2}{2}\right) \cdot \ldots \cdot \left(\frac{3}{2}\right) \frac{(3n/2)!}{(3n/2)!} = (3n - 1)!!$$

such matchings. Let $\mathcal{F}_3(n)$ denote the collection of all $(3n - 1)!!$ possible pairings on $W_n$. We project each pairing $F \in \mathcal{F}_3(n)$ to a multigraph $\pi(F)$ on the vertex set $V_n$ by ignoring the second coordinate. Then $\pi(F)$ is a 3-regular multigraph (which may or may not contain loops and multi-edges). Let $\pi(\mathcal{F}_3(n)) = \{\pi(F) : F \in \mathcal{F}_3(n)\}$ be the set of 3-regular multigraphs on $V_n$. By definition,

Each simple graph $G \in \pi(\mathcal{F}_3(n))$ corresponds to $(3!)^n$ distinct pairings in $\mathcal{F}_3(n)$.

We will call the elements of $V_n - \text{vertices}$, and of $W_n - \text{points}$.

Definition 2. Let $G_g(n)$ be the set of all cubic graphs with vertex set $V_n = [n]$ and girth at least $g$ and $G_{g'}(n) = \{F \in \mathcal{F}_3(n) : \pi(F) \in G_g(n)\}$.

We will use the following result:

Theorem 3 (Wormald [22], Bollobás [3]). For each fixed $g \geq 3$,

$$\lim_{n \to \infty} \frac{|G_g(n)|}{|\mathcal{F}_3(n)|} = \exp \left\{ - \sum_{k=1}^{g-1} \frac{2^{k-1}}{k} \right\}.$$  

(2)

Remark. When we say that a pairing $F$ has a multigraph property $\mathcal{A}$, we mean that $\pi(F)$ has property $\mathcal{A}$.

Since dealing with pairings is simpler than working with labeled simple regular graphs, we need the following well-known consequence of Theorem 3.

Corollary 4 ([20] (Corollary 1.1), [16] (Theorem 9.5)). For fixed $g \geq 3$, any property that holds for $\pi(F)$ for almost all pairings $F \in \mathcal{F}_3(n)$ also holds for almost all graphs in $G_g(n)$.

Proof. Suppose property $\mathcal{A}$ holds for $\pi(F)$ for almost all $F \in \mathcal{F}_3(n)$. Let $\mathcal{H}(n)$ denote the set of graphs in $G_g(n)$ that do not have property $\mathcal{A}$ and $\mathcal{H}'(n) = \{F \in \mathcal{F}_3(n) : \pi(F) \in \mathcal{H}(n)\}$. Let $\mathcal{B}(n)$ denote the set of pairings $F \in \mathcal{F}_3(n)$ such that $\pi(F)$ does not have property $\mathcal{A}$. Then $\mathcal{H}'(n) \subseteq \mathcal{B}(n)$. Hence by the choice of $\mathcal{A}$,

$$\frac{|\mathcal{H}'(n)|}{|\mathcal{F}_3(n)|} \leq \frac{|\mathcal{B}(n)|}{|\mathcal{F}_3(n)|} \to 0 \quad \text{as } n \to \infty.$$  

(3)

By (1), we have

$$\frac{|\mathcal{H}(n)|}{|G_g(n)|} = \frac{|\mathcal{H}(n)|}{|\mathcal{H}'(n)|} \cdot \frac{|\mathcal{H}'(n)|}{|G_g(n)|} = \frac{1}{(3!)^n} \frac{|\mathcal{H}(n)|}{|G_g(n)|} \cdot (3!)^n = \frac{|\mathcal{H}(n)|}{|G_g(n)|}.$$
Furthermore,

\[ \frac{|\mathcal{H}(n)|}{|G(n)|} = \frac{|\mathcal{H}(n)|}{|\mathcal{F}(n)|} \cdot \frac{|\mathcal{F}(n)|}{|G(n)|}. \tag{4} \]

By (3) and Theorem 3, the right-hand side of (4) tends to 0 as \( n \) tends to infinity. 

3. Bounds for \( c_1, c_2, \ldots \)

We will use the following theorem of McKay [20].

**Theorem 5 (McKay [20]).** For every \( \epsilon > 0 \), there exists an \( N > 0 \) such that for each \( n > N \),

\[ |\{F \in \mathcal{F}(n) : c_1(\pi(F)) > 0.45537n\}| < \epsilon \cdot (3n - 1)!! \]

**Definition 6.** A 3-regular tree is a tree such that each vertex has degree 3 or 1. A \((3, k, a)\)-tree is a rooted 3-regular tree \( T \) with root \( a \) of degree 3 such that the distance in \( T \) from each of the leaves to \( a \) is \( k \) (see Fig. 1).

**Definition 7.** For a positive integer \( s \) and a vertex \( a \) in a graph \( G \), the ball \( B_C(a, s) \) in \( G \) of radius \( s \) with center \( a \) is \( \{v \in V(G) : d_C(v, a) \leq s\} \), where \( d_C(v, a) \) denotes the distance in \( G \) from \( v \) to \( a \).

We first prove simple bounds on \( c_{2k}(G) \) and \( c_{2k+1}(G) \) when \( G \in \mathcal{G}_{2k+2}(n) \).

**Lemma 8.** Let \( j \) be a fixed positive integer and \( n > g \geq 2j + 2 \). Then for every \( G \in \mathcal{G}_g(n) \),

(i) \( c_j(G) \leq \frac{n}{3 \cdot 2^j - 2} \),

and

(ii) \( c_{2j+1}(G) \leq \frac{c_j(G)}{2^j - 1} \).

**Proof.** (i) Let \( C_j \) be a \( 2j \)-independent set in \( G \) with \( |C_{2j}| = c_{2j}(G) \). Since the distance between any distinct \( a, b \in C_{2j} \) is at least \( 2j + 1 \), the balls \( B_C(a, j) \) for all distinct \( a \in C_{2j} \) are disjoint. Moreover, since \( g \geq 2j + 2 \), each ball \( B_C(a, j) \) induces a \((3, j, a)\)-tree \( T_a \), and hence has \( 1 + 3 + 3 \cdot 2 + 3 \cdot 2^2 + \cdots + 3 \cdot 2^{j-1} = 3 \cdot 2^j - 2 \) vertices. This proves (i).

(ii) Let \( C_{2j+1} \) be a \((2j + 1)\)-independent set in \( G \) with \( |C_{2j+1}| = c_{2j+1}(G) \). As in the proof of (i), the balls \( B_C(a, j) \) for distinct \( a \in C_{2j} \) are disjoint, and each \( B_C(a, j) \) induces a \((3, j, a)\)-tree \( T_a \). But in this case, in addition, the balls with centers in distinct vertices of \( C_{2j+1} \) are at distance at least 2 from each other. Let \( S_i \) be the set of vertices in \( T_a \) at distance \( i \) from \( a \). Then \( |S_0| = 1 \), and for each \( 1 \leq i \leq j \), \( |S_i| = 3 \cdot 2^{i-1} \). If follows that the set \( I_a = \bigcup_{i=0}^{j/2} S_{j/2} \) is independent, and

\[ |I_a| = \sum_{i=0}^{j/2} |S_{j/2-i}| = 2^{j+1} - 1. \]

Therefore \( I := \bigcup_{a \in C_{2j+1}} I_a \) is an independent set in \( G \) and \(|I| = (2^{j+1} - 1)c_{2j+1}(G) \). This implies (ii). 

**Lemma 9.** Let \( k \) be a fixed positive integer and \( x \) be a real number with \( 0 < x < \frac{1}{3 \cdot 2^k - 2} \). The number of pairings \( F \in \mathcal{G}'_{2k+2}(n) \) such that \( \pi(F) \) has a \( 2k \)-independent vertex set of size \( xn \) is at most

\[ q(n, k, x) := \binom{n}{xn} \cdot \left( 3n - (6 \cdot 2^k - 6)xn - 1 \right)! \prod_{i=0}^{k-1} \left( \frac{1 - (3 \cdot 2^i - 2)x}{3 \cdot 2^i} \right) ! \cdot (3 \cdot 2^k xn)! \cdot 3^{3 \cdot 2^k xn}. \]

**Proof.** To prove the lemma, we will show that the total number of \( 2k \)-independent sets of size \( xn \) in \( \pi(F) \) over all \( F \in \mathcal{G}'_{2k+2}(n) \) does not exceed \( q(n, k, x) \). Below we describe a procedure of constructing for every \( C \subset [n] \) with \( |C| = xn \) all pairings \( F \in \mathcal{G}'_{2k+2}(n) \) for which \( C \) is \( 2k \)-independent in \( \pi(F) \). Not every obtained pairing will be in \( \mathcal{G}'_{2k+2}(n) \), but every \( F \in \mathcal{G}'_{2k+2}(n) \) such that \( C \) is a \( 2k \)-independent set in \( \pi(F) \) will be a result of this procedure:

1. We choose a vertex set \( C \) of size \( xn \) from \([n]\). There are \( \binom{n}{xn} \) ways to do it.
Corollary 10. Let $g$ and $2. In order $C$ to be $2k$-independent and $\pi(F)$ to have girth at least $2k+2$, all the balls of radius $k$ with the centers in $C$ must be disjoint, and for each $a \in C$, the ball $B_{\pi(F)}(a, k)$ must induce a $(3, k, a)$-tree. Thus, we have $(\frac{1-x^n}{3x})$ ways to choose the neighbors of $C$, call it $N(C)$, $(3x)!$ ways to determine which vertex in $N(C)$ will be the neighbor for each point in $\pi^{-1}(C)$, and $3^{3x}$ ways to decide which point of each vertex in $N(C)$ is adjacent to the corresponding point in $\pi^{-1}(C)$. Each vertex of $N(C)$ will have $2$ free points left at this moment, and in total the set $\pi^{-1}(N(C))$ has now $2 \cdot 3x = 6x$ free points.

3. Similarly to the previous step, consecutively for $i = 1, 2, \ldots, k-1$, we will decide which vertices and points are in the set $\pi^{-1}(N^{i+1}(C))$ of the vertices at distance $i$ from $C$, as follows. Before the $i$th iteration, we have $3x \cdot 2^i n$ free points in the $3x \cdot 2^{i-1} n$ vertices of $\pi^{-1}(N^i(C))$, and

$$|C \cup N^1(C) \cup \cdots \cup N^i(C)| = xn (1 + 3(1 + 2 + \cdots + 2^{i-1})) = (3 \cdot 2^i - 2)xn.$$ 

We choose $3x \cdot 2^i n$ vertices out of the remaining $(1 - (3 \cdot 2^i - 2)x)n$ vertices to include into $N^{i+1}(C)$, then we have $(3x \cdot 2^i)!$ ways to determine which vertex in $N^{i+1}(C)$ will be the neighbor for each free point in $\pi^{-1}(N^i(C))$, and $3^{3x \cdot 2^i}$ ways to decide which point of each vertex in $N^{i+1}(C)$ is adjacent to the corresponding point in $\pi^{-1}(N^i(C))$.

4. Finally, there are $3n - (6 \cdot 2^k - 6)xn$ free points left and we have $(3n - (6 \cdot 2^k - 6)n - 1)!$ ways to pair them. Multiplying the quantities in 1–4 above, we obtain $q(n, k, x)$. This proves the bound. ■

In the proofs below we will use Stirling’s formula: For every $n \geq 1$,

$$\sqrt{2\pi n} \left(\frac{n}{e}\right)^n \leq n! \leq \sqrt{2\pi n} \left(\frac{n}{e}\right)^n e^{1/12n}. \quad (5)$$

Corollary 10. Let $g \geq 22$ be fixed. For every $\epsilon > 0$, there exists an $N > 0$ such that for each $n > N$,

$$|\{G \in G_g(n) : c_2(G) > 0.236n =: b_2n\}| < \epsilon \cdot |G_g(n)|, \quad (6)$$

$$|\{G \in G_g(n) : c_4(G) > 0.082n =: b_4n\}| < \epsilon \cdot |G_g(n)|, \quad (7)$$

$$|\{G \in G_g(n) : c_6(G) > 0.03n =: b_6n\}| < \epsilon \cdot |G_g(n)|, \quad (8)$$

$$|\{G \in G_g(n) : c_8(G) > 0.011n =: b_8n\}| < \epsilon \cdot |G_g(n)|, \quad (9)$$

and

$$|\{G \in G_g(n) : c_{10}(G) > 0.004n =: b_{10}n\}| < \epsilon \cdot |G_g(n)|. \quad (10)$$
Proof. By Lemma 9,
\[ q(n, k, x) = \binom{n}{x^n} \cdot ((3n - (6 \cdot 2^k - 6)xn - 1)!!) \prod_{i=0}^{k-1} \left( \frac{(1 - (3 \cdot 2^i - 2)xn)^n}{3 \cdot 2^{i \cdot xn}} \right) \cdot ((3 \cdot 2^{x \cdot xn})^{(3 \cdot 2^{2 \cdot xn})}) \]
\[ = \frac{(3n - (6 \cdot 2^k - 6)xn - 1)!! \cdot n!}{(xn)! \cdot ((1 - x)n)!} \cdot 3^{3 \cdot xn + 6xn + \cdots + 3 \cdot 2^{k-1}xn} \]
\[ = \frac{((1 - x)n)! \cdot (3xn)!}{(3xn)! \cdot ((1 - 4x)n)! \cdot (6xn)! \cdot ((1 - 10x)n)! \cdots ((1 - (3 \cdot 2^k - 2)xn)! \cdot ((1 - (3 \cdot 2^k - 2)xn)!} \]
\[ = \frac{(3n - (6 \cdot 2^k - 6)xn - 1)!! \cdot n!}{(xn)! \cdot ((1 - (3 \cdot 2^k - 2)xn)! \cdot 3^{3 \cdot 2^{2-3}xn}.} \]

We know that
\[ (3n - 1)!! = \frac{(3n)!!}{3n} \geq \frac{\sqrt{3n}}{3n} \]
and
\[ (3n - (6 \cdot 2^k - 6)xn - 1)!! \leq \sqrt{3n - (6 \cdot 2^k - 6)xn}. \]

Therefore,
\[ q(n, k, x) \leq (3n) \cdot \left( \frac{(3n - (6 \cdot 2^k - 6)xn)!}{(3n)!} \right)^{\frac{1}{2}} \cdot \frac{n!}{(xn)!! \cdot ((1 - (3 \cdot 2^k - 2)xn)!} \cdot 3^{3 \cdot 2^{2-3}xn}. \]

Using Stirling’s formula (5), we have
\[ q(n, k, x) = O(n^2) \cdot \frac{(\frac{3n}{2})^{\frac{1}{2}(3n - (6 \cdot 2^k - 6)xn)}}{(\frac{3n}{2})^{\frac{1}{2}n}} \cdot \frac{(\frac{(1 - (2k+1 - 2)x)!}{\binom{1 - (3 \cdot 2^k - 2)x!}{1 - (3 \cdot 2^k - 2)x!}})^n}{x^n(1 - (3 \cdot 2^k - 2)x)^{1 - (3 \cdot 2^2 - 2)x}}. \]

Let
\[ f(x, k) = \frac{(1 - (2k+1 - 2)x)^{1 - (3 \cdot 2^k - 2)x}}{x^n(1 - (3 \cdot 2^k - 2)x)^{1 - (3 \cdot 2^2 - 2)x}}, \]
so that
\[ q(n, k, x) = O(n^2) \cdot f(x, k). \]

By plugging \( x = 0.236 \) and \( k = 1 \) into (11) (using a computer or a good calculator), we see that \( 0 < f(0.236, 1) < 0.9964. \)

Similarly, by plugging the corresponding values of \( x \) and \( k \) into (11), one can check that \( 0 < f(0.082, 2) < 0.9977, 0 < f(0.03, 3) < 0.9981, 0 < f(0.011, 4) < 0.996, \) and \( 0 < f(0.004, 5) < 0.995. \) Thus repeating the argument of the previous paragraph, we obtain that (7), (8), (9), (10) also hold. 

Lemma 11. Let \( k \) be a fixed positive integer and \( 0 < x < \frac{0.455377}{2^k - 1}. \) The number of pairings \( F \in \mathcal{G}_{2k+2}(n) \) such that \( \pi(F) \) has a \((2k + 1)\)-independent vertex set of size \( xn \) is at most
\[ r(n, k, x) := \binom{n}{x^n} \cdot (3n - (3 \cdot 2^k - 2)xn)! \cdot (3n - (4 \cdot 2^k - 2)xn - 1)!!\]
\[ \times \frac{(1 - (3 \cdot 2^i - 2)xn)!}{3 \cdot 2^{i \cdot xn}} \cdot (3 \cdot 2^{x \cdot xn})^{(3 \cdot 2^{2 \cdot xn})} \]
\[ \times \prod_{i=0}^{k-1} \left( \frac{(1 - (3 \cdot 2^i - 2)xn)!}{3 \cdot 2^{i \cdot xn}} \right) \cdot (3 \cdot 2^{x \cdot xn})^{(3 \cdot 2^{2 \cdot xn})}. \]

Proof. We will show that the total number of \((2k + 1)\)-independent sets of size \( xn \) in \( \pi(F) \) over all \( F \in \mathcal{G}_{2k+2}(n) \) does not exceed \( r(n, k, x) \). Below we describe a procedure of constructing for every set \( C \) of size \( xn \) in \([n]\) all pairings in \( \mathcal{G}_{2k+2}(n) \) for
which C is \((2k + 1)\)-independent. Not every obtained pairing will be in \(G'_{2k+2}(n)\), but every \(F \in G'_{2k+2}(n)\) such that C is a \((2k + 1)\)-independent set in \(\pi(F)\) will be a result of this procedure:

1. We choose a vertex set C of size \(xn\) from \([n]\). There are \(\binom{n}{xn}\) ways to do it.
2. In order C to be \((2k + 1)\)-independent and \(\pi(F)\) to have girth at least \(2k + 2\), all the balls of radius \(k\) with the centers in C must be disjoint, and for each \(a \in C\), the ball \(B_{\pi(F)}(a, k)\) must induce a \((3, k, a)\)-tree. Thus, we have \(\binom{n}{3xn}\) ways to choose the neighbors of C, call it \(N(C)\), \(3xn!\) ways to determine which vertex in \(N(C)\) will be the neighbor for each point in \(\pi^{-1}(C)\), and \(3^{3xn}\) ways to decide which point of each vertex in \(N(C)\) is adjacent to the corresponding point in \(\pi^{-1}(C)\). Each vertex of \(N(C)\) will have \(2\) free points left at this moment, and in total the set \(\pi^{-1}(N(C))\) has now \(2 \cdot 3xn = 6xn\) free points.
3. Similarly to the previous step, consecutively for \(i = 1, 2, \ldots, k - 1\), we will decide which vertices and points are in the set \(\pi^{-1}(N_i^{i+1}(C))\) of the vertices at distance \(i\) from C, as follows. Before the \(i\)th iteration, we have \(3x \cdot 2^n\) free points in the \(3x \cdot 2^{i-1}\) vertices of \(\pi^{-1}(N_i^{i}(C))\), and

\[
|C \cup N_1^1(C) \cup \cdots \cup N_i^i(C)| = xn \left(1 + 3(1 + 2 + \cdots + 2^{i-1})\right) = (3 \cdot 2^i - 2)xn.
\]

We choose \(3x \cdot 2^n\) vertices out of the remaining \((1 - (3 \cdot 2^i - 2)xn)\) vertices to include into \(N_i^{i+1}(C)\), then we have \((3x \cdot 2^n)!\) ways to determine which vertex in \(N_i^{i+1}(C)\) will be the neighbor for each free point in \(\pi^{-1}(N_i^{i}(C))\), and \(3^{3x \cdot 2^n}\) ways to decide which point of each vertex in \(N_i^{i+1}(C)\) is adjacent to the corresponding point in \(\pi^{-1}(N_i^{i}(C))\).
4. Let \(N_0^0(C) := C\) and \(S := \cup_{i=0}^k N_i^{i}(C)\). In order the distance between each pair of vertices in C to be at least \(2k + 2\), \(N_k(C)\) has to be an independent set. Therefore, each of the \(3x \cdot 2^n\) free points in the \(3x \cdot 2^{k-1}\) vertices of \(\pi^{-1}(N_k(C))\) has to be paired with one of the remaining \(3(n - (3 \cdot 2^k - 2)xn)\) free points of \(\pi^{-1}(\{n\} - S)\) and we have

\[
\begin{align*}
(3(n - (3 \cdot 2^k - 2)xn))! \\
(3(n - (4 \cdot 2^k - 2)xn))!
\end{align*}
\]

to do that.
5. Finally, there are \(3(n - (4 \cdot 2^k - 2)xn)\) free points left and we have \((3(n - (4 \cdot 2^k - 2)xn) - 1)!!\) ways to pair them.

The product of the numbers of choices in the above steps 1–5 equals \(r(n, k, x)\), which proves the lemma. ■

**Corollary 12.** Let \(g \geq 24\) be fixed. For every \(\varepsilon > 0\), there exists an \(N > 0\) such that for each \(n > N\),

\[
||G \in G_g(n) : c_3(G) > 0.1394n := b_3n|| < \varepsilon \cdot |G_g(n)|, \tag{14}
\]

\[
||G \in G_g(n) : c_5(G) > 0.05n := b_5n|| < \varepsilon \cdot |G_g(n)|, \tag{15}
\]

\[
||G \in G_g(n) : c_7(G) > 0.0182n := b_7n|| < \varepsilon \cdot |G_g(n)|, \tag{16}
\]

\[
||G \in G_g(n) : c_9(G) > 0.0063n := b_9n|| < \varepsilon \cdot |G_g(n)|, \tag{17}
\]

and

\[
||G \in G_g(n) : c_{11}(G) > 0.0022n := b_{11}n|| < \varepsilon \cdot |G_g(n)|. \tag{18}
\]

**Proof.** By **Lemma 11**, \(r(n, k, x) = \binom{n}{xn} \cdot (3(n - (3 \cdot 2^k - 2)xn))! \cdot (3(n - (4 \cdot 2^k - 2)xn) - 1)!!

\[
\times \prod_{i=0}^{k-1} \binom{\left(1 - (3 \cdot 2^i - 2)xn\right)!!}{3 \cdot 2^i xn} \cdot (3 \cdot 2^i xn)! \cdot 3 \cdot 2^{i+1} xn
\]

\[
= \frac{(3(n - (3 \cdot 2^k - 2)xn))! \cdot (3(n - (4 \cdot 2^k - 2)xn) - 1)!!}{(3(n - (4 \cdot 2^k - 2)xn))!} \cdot \frac{n!}{(xn)! \cdot ((1 - x)n)!} \cdot 3 \cdot 2^{k+1} xn
\]

\[
\cdot \binom{(1 - x)n)! \cdot (3xn)!}{(3xn)! \cdot ((1 - 4x)n)! \cdot (6xn)!} \cdot \binom{(1 - 4x)n)! \cdot (6xn)! \cdot (1 - 10x)n)!}{(3 \cdot 2^{k-1} xn)! \cdot ((1 - (3 \cdot 2^k - 2)xn)! \cdot (3 \cdot 2^{k-1} xn)! \cdot ((1 - (3 \cdot 2^k - 2)xn)!}
\]

\[
= \frac{(3(n - (3 \cdot 2^k - 2)xn))! \cdot (3(n - (4 \cdot 2^k - 2)xn) - 1)!!}{(3(n - (4 \cdot 2^k - 2)xn))!} \cdot \frac{n!}{(xn)! \cdot ((1 - (3 \cdot 2^k - 2)xn)! \cdot 3 \cdot 2^{k-1} xn}.
\]

By the definition of the double factorial,
\[(3n - 1)!! \geq \frac{(3n)!!}{3n} \geq \frac{\sqrt{3n!}}{3n}\]
and
\[(3(n - (4 \cdot 2^k - 2)xn) - 1)!! \leq \sqrt{(3(n - (4 \cdot 2^k - 2)xn))!!}.
\]
Therefore,
\[
\begin{align*}
\frac{r(n, k, x)}{(3n - 1)!!} &\leq (3n) \cdot \left(\frac{(3(n - (4 \cdot 2^k - 2)xn))!!}{(3n)!}\right)^\frac{1}{2} \cdot \frac{(3(n - (3 \cdot 2^k - 2)xn))!!}{(3(n - (4 \cdot 2^k - 2)xn))!!} \\
&\quad \cdot \frac{n!}{(xn)! \cdot ((1 - (3 \cdot 2^k - 2)x)n)!}. \cdot 3^{(3 \cdot 2^k - 3)xn}.
\end{align*}
\]
By Stirling’s formula (5),
\[
\begin{align*}
\frac{r(n, k, x)}{(3n - 1)!!} &= O(n^3) \cdot \left(\frac{n}{\sqrt{2\pi n}}\right)^3 \cdot \frac{(n - (4 \cdot 2^k - 2)xn)!}{(n - (3 \cdot 2^k - 2)xn)!} \cdot \frac{n!}{(xn)! \cdot ((1 - (3 \cdot 2^k - 2)x)n)!} \\
&\quad \cdot \left(\frac{1 - (3 \cdot 2^k - 2)x\cdot 2 - (6\cdot 2^k - 4)x^2}{x^3(1 - (4 \cdot 2^k - 2)x)^{1.5 - (6\cdot 2^k - 3)x}}\right)^n
\end{align*}
\]
Let
\[
h(x, k) = \frac{(1 - (3 \cdot 2^k - 2)x\cdot 2 - (6\cdot 2^k - 4)x^2)}{x^3(1 - (4 \cdot 2^k - 2)x)^{1.5 - (6\cdot 2^k - 3)x}},
\]
so that
\[
\frac{r(n, k, x)}{|F_3(n)|} = O(n^3) = O(n^3)(h(x, k))^n. \tag{21}
\]
By plugging \(x = 0.1394\) and \(k = 1\) into (20) (using a computer or a calculator), we see that \(0 < h(0.1394, 1) < 0.9974\). Since \(h(x, 1)\) is a smooth function for \(0 < x < 1\), there exists \(\nu_1\) such that \(h(x, 1) < 0.9974\) for all \(x \in [0.1394 - \nu_1, 0.1394]\). If \(n > 1/\nu_1\), then there exists an \(x_1 = x_1(n) \in [0.1394 - \nu_1, 0.1394]\) such that \(x_1n\) is an integer. By (21),
\[
\frac{r(n, 1, x_1n)}{|F_3(n)|} = O(n^3)(0.9974)^n \to 0 \quad \text{as} \ n \to \infty.
\]
By the definition of \(r(n, k, x)\), (2) and Corollary 4, this implies (14).

Similarly, by plugging the corresponding values of \(x\) and \(k\) into (20), one can check that \(0 < h(0.05, 2) < 0.9985, 0 < h(0.0182, 3) < 0.9973, 0 < h(0.0063, 4) < 0.9986,\) and \(0 < h(0.0022, 5) < 0.9979\). Thus repeating the argument of the previous paragraph, we obtain that (15), (16), (17), (18) also hold.

4. Bound on \(|C_1 \cup C_2 \cup C_4|\)

**Definition 13.** For a graph \(G\), let \(c_{1,2,4}(G)\) be the maximum size of \(|C_1 \cup C_2 \cup C_4|\), where \(C_1, C_2\) and \(C_4\) are disjoint subsets of \(V(G)\) such that \(C_i\) is \(i\)-independent for all \(i \in \{1, 2, 4\}\).

In this section we prove an upper bound on \(c_{1,2,4}(G)\) in terms of \(c_1(G)\) for cubic graphs \(G\) of girth at least 9. For every vertex \(a\) in such a graph \(G\), the ball \(B_2(a, 2)\) induces a \((3, 2, a)\)-tree \(T_a\). When handling such a tree \(T_a\), we will use the following notation (see Fig. 2):
\[
V(T_a) = \{a\} \cup N_1(a) \cup N_2(a), \quad \text{where} \quad N_1(a) = \{a_1, a_2, a_3\}, \quad N_2(a) = \{a_{1,1}, a_{1,2}, a_{2,1}, a_{2,2}, a_{3,1}, a_{3,2}\},
\]
and
\[
E(T) = \{aa_1, aa_2, aa_3, a_1a_{1,1}, a_1a_{1,2}, a_2a_{2,1}, a_2a_{2,2}, a_3a_{3,1}, a_3a_{3,2}\}.
\]
Lemma 14. Let \( G \) be an \( n \)-vertex cubic graph with girth at least 9 and
\[
c_1(G) < 0.456n. \tag{22}
\]
Then \( c_{1,2,4}(G) \leq 0.7174n =: b_{1,2,4}n. \)

Proof. Let \( G \) satisfy the conditions of the lemma, and let \( C_1, C_2 \) and \( C_4 \) be disjoint subsets of \( V(G) \) such that \( C_i \) is \( i \)-independent for \( i \in \{1, 2, 4\} \) and \( |C_1 \cup C_2 \cup C_4| = c_{1,2,4}(G) \).

The idea of the proof uses the fact that for a typical vertex \( a \in C_4 \), the tree \( T_a \) contains several vertices not in \( C_1 \cup C_2 \). For example, each vertex in \( C_4 \) has at most one neighbor in \( C_2 \). Also for distinct \( a_1, a_2 \in C_4 \), the trees \( T_{a_1} \) and \( T_{a_2} \) are vertex-disjoint. For more accurate counting, we need a couple of new notions. Let \( Q \) be the set of vertices in \( C_1 \) that do not have neighbors in \( C_2 \), and \( L \) be the set of edges in \( G \setminus C_1 \setminus C_2 \). For brevity, the vertices in \( Q \) will be called \( Q \)-vertices, and the edges in \( L \) will be called \( L \)-edges. Let \( s = |C_1| + |C_2| \). It will turn out that \( q + \ell \) is a convenient parameter helping to bound \( |C_4| \) in terms of \( s \) and \( |C_2| \). We will prove the lemma in a series of claims. Our first claim is:

\[
s < 0.652n. \tag{23}
\]

To show \( (23) \), we count the edges connecting \( C_1 \cup C_2 \) with \( \overline{C_1 \cup C_2} \) in two ways:
\[
3(n - s) - 2\ell = e[\overline{C_1 \cup C_2}, \overline{C_1 \cup C_2}] = 3s - 2(|C_1| - q). \tag{24}
\]

Solving for \( s \), we get
\[
s > \frac{n}{3} - \frac{1}{2}(\ell - |C_1| + q). \tag{25}
\]

Indeed, let \( 0 \leq j \leq 2 \) and \( a \in S_j \). If a vertex \( a \in N_1(a) \) is not in \( C_1 \cup C_2 \), then either \( a \in Q \) or \( aa \in L \). Thus, by the definition of \( S_j \), \(|N_1(a) \cap ((C_1 \cup C_2) - \overline{Q})| \geq 3 - j \). Since each \( a_i \in (C_1 \cup C_2) - Q \) either is in \( C_2 \) or has a neighbor in \( C_2 \cap \{a_1, a_2\} \), we get at least \( 3 - j \) vertices in \( C_2 \setminus V(T_a) \). This proves \( (25) \).

For \( 0 \leq j \leq 2 \), let \( |S_j| = \alpha_jn \), and let \( |U| = \beta n \). Then
\[
(\alpha_1 + \alpha_2 + \alpha_3 + \beta)n = |C_4|. \tag{26}
\]

By the definition of 4-independent sets, for all \( a \in C_4 \) the balls \( B_C(a, 2) \) are disjoint and not adjacent to each other. For \( 0 \leq j \leq 2 \) and every \( a \in S_j \), the tree \( T_a \) contributes \( j \) to \( \ell + q \), and for every \( a \in U, T_a \) contributes at least \( 3 \) to \( \ell + q \). Therefore
\[
\alpha_1n + 2\alpha_2n + 3\beta n \leq \ell + q. \tag{27}
\]
Also, (25) yields a lower bound on $|C_2|$:

$$3\alpha_0n + 2\alpha_1n + \alpha_2n \leq |C_2|.$$  \hfill (28)

Now (26), (27), and (28) yield

$$3|C_4| = (\alpha_1n + 2\alpha_2n + 3\beta n) + (3\alpha_0n + 2\alpha_1n + \alpha_2n) \leq \ell + q + |C_2|. \quad \hfill (29)$$

On the other hand, by (24)

$$2(\ell + q) = 3n - 6s + 2|C_1| = 3n - 4s - 2|C_2|,$$

so $2(\ell + q + |C_2|) = 3n - 4s$. Comparing with (29), we get

$$|C_4| \leq \frac{3n - 4s}{6} = \frac{3n + 2s}{6} - s.$$

Hence by the definition of $s$ and (23),

$$|C_1 \cup C_2 \cup C_4| = |C_4| + s \leq \frac{3n + 2s}{6} \leq \frac{n}{2} + \frac{0.652n}{3} \leq 0.7174n.$$  \hfill \blacksquare

5. Proof of Theorem 1

For each fixed integer $k \geq 12$ and $g \geq 2k + 2$, let $J := \{3, 5, 6, 7, \ldots, 11\}$ and

$$B_g(n) = \left\{ G \in \mathcal{G}_g(n) : c_{1,2,4}(G) + \sum_{j \in J} c_j(G) > 0.9785n \right\}.$$

\textbf{Lemma 15}. Let $k \geq 12$ be a fixed integer and $g \geq 2k + 2$. For every $\varepsilon > 0$, there exists an $N = N(\varepsilon) > 0$ such that for each $n > N$,

$$|B_g(n)| < \varepsilon \cdot |\mathcal{G}_g(n)|.$$  \hfill (31)

\textbf{Proof}. Let $\varepsilon > 0$ be given. By \textbf{Lemma 14}, \textbf{Theorem 5}, and \textbf{Corollary 4}, there exists an $N_{1,2,4} > 0$ such that for each $n > N_{1,2,4}$,

$$|\{ G \in \mathcal{G}_g(n) : c_{1,2,4}(G) > b_{1,2,4}n \}| < \frac{\varepsilon}{10} \cdot |\mathcal{G}_g(n)|.$$

Let

$$M_{1,2,4}(n) := \{ G \in \mathcal{G}_g(n) : c_{1,2,4}(G) > b_{1,2,4}n \}.$$

For each $j \in J$ and the constants $b_j$ defined in \textbf{Corollaries 10} and \textbf{12}, let

$$M_j(n) := \{ G \in \mathcal{G}_g(n) : c_j(G) > b_jn \}.$$

Let

$$B_g'(n) = \left\{ G \in \mathcal{G}_g(n) : c_{1,2,4}(G) + \sum_{j \in J} c_j(G) > 0.9785n \right\},$$

and

$$B_g''(n) = \left\{ G \in \mathcal{G}_g(n) : c_1(G) > 0.45537n \right\}.$$

If $G \in B_g'(n)$, then

$$G \in M_{1,2,4}(n) \cup \bigcup_{j \in J} M_j(n),$$

because $b_{1,2,4}n + \sum_{j \in J} b_jn = 0.9785n$ and $c_{1,2,4} + \sum_{j \in J} c_j > 0.9785n$.

\textbf{Corollaries 10} and \textbf{12} imply that for each $j \in J$, there exists an $N_j > 0$ such that for each $n > N_j$,

$$|\{ G \in \mathcal{G}_g(n) : c_j(G) > b_jn \}| < \frac{\varepsilon}{10} \cdot |\mathcal{G}_g(n)|.$$

By \textbf{Theorem 5}, there exists an $N_1 > 0$ such that for each $n > N_1$, $|B_g''(n)| < \frac{\varepsilon}{10} \cdot |\mathcal{G}_g(n)|$.

Let $N = \max\{N_{1,2,4}, N_1, N_3, N_5, N_6, \ldots, N_{11}\}$. By the definition of $N$, for each $n > N$,

$$|B_g'(n)| + |B_g''(n)| < (1 + |J| + 1)\frac{\varepsilon}{10} \cdot |\mathcal{G}_g(n)| = \varepsilon \cdot |\mathcal{G}_g(n)|.$$

\hfill (32)
Every graph $G \in \mathcal{G}_g(n) \setminus \mathcal{B}^c_g(n)$ satisfies $c_1(G) \leq 0.45537n$. Using this, Lemma 8(ii) implies that such a graph $G$ satisfies

$$
\sum_{j=6}^{\lceil k/2 \rceil - 1} c_{2j+1}(G) < \frac{\lceil k/2 \rceil - 1}{2} \leq \frac{0.45537n}{2^{2+1} - 1} \leq \frac{0.45537n}{127} \cdot \sum_{i=0}^{\infty} \frac{1}{2^i} = \frac{2 \cdot 0.45537n}{127}.
$$

It follows that $\mathcal{B}^c_g(n) \subset \mathcal{B}^c_g(n) \cup \mathcal{B}^c_g(n)$. Thus (32) implies (31). $\blacksquare$

Now we are prepared to prove our main result.

**Proof of Theorem 1.** Let $k \geq 12$ be a fixed integer and $g \geq 2k + 2$. We need to show that for every $\varepsilon > 0$, there exists an $N > 0$ such that for each $n > N$,

$$
|\{G \in \mathcal{G}_g(n) : \chi_p(G) \leq k\}| < \varepsilon \cdot |\mathcal{G}_g(n)|. \tag{33}
$$

Let $\varepsilon > 0$ be given and $G \in \mathcal{G}_g(n)$ satisfy $\chi_p(G) \leq k$. Then there is a partition of $V(G)$ into $C_1, C_2, \ldots, C_k$ such that for each $i = 1, 2, \ldots, k$, $C_i$ is $i$-independent. In particular, $|C_1| + |C_2| + \cdots + |C_k| = n$. By Lemma 8(i),

$$
\sum_{j=6}^{\lceil k/2 \rceil} |C_{2j}| < \sum_{k=6}^{\infty} \frac{n}{3 \cdot 2^k} < \frac{n}{190} \sum_{k=0}^{\infty} \frac{1}{2^k} = \frac{n}{95}. \tag{34}
$$

Since $n - \frac{n}{95} > 0.9785n + \frac{2 \cdot 0.45537n}{127}$, this implies that $G \in \mathcal{B}_g(n)$, where $\mathcal{B}_g(n)$ is defined by (30). Thus, Lemma 15 implies (33). $\blacksquare$

6. Concluding remarks

1. It seems that with more sophisticated calculations, one can prove the claim of Theorem 1 not only for almost all cubic graphs with girth at least $2k + 2$, but at almost all cubic $n$-vertex graphs. But we cannot prove (and maybe this is not true) that for each $k$ every cubic graph $G$ with sufficiently large (with respect to $k$) girth satisfies $\chi_p(G) > k$.

2. Our approach does not yield cubic planar graphs with high packing chromatic number. Gastineau, Holub and Togni [13] showed that $\chi_p(G)$ is bounded for graphs $G$ in some subclasses of cubic outerplanar graphs.

Acknowledgments

We thank the referees for their valuable comments. Research of first author is partially supported by NSF Grant DMS-1500121, Arnold O. Beckman Research Award (UIUC Campus Research Board 15006) and by the Langan Scholar Fund (UIUC). Research of second author is supported in part by NSF grant DMS-1266016 and by grants 15-01-05867 and 16-01-00499 of the Russian Foundation for Basic Research.

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