Equitable colorings
of sparse graphs

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Based on joint work with H. Kierstead
**Definition**

An **equitable coloring** of a graph is a proper vertex coloring such that the sizes of every two color classes differ by at most 1.
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Let $H(n, k)$ denote the $n$-vertex graph every of whose $k$ components is either a $\left\lfloor \frac{n}{k} \right\rfloor$-clique or a $\left\lceil \frac{n}{k} \right\rceil$-clique.

An $n$-vertex graph $G$ has an equitable $k$-coloring if and only if $G$ packs with $H(n, k)$.
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An $n$-vertex graph $G$ has an equitable $k$-coloring $\iff$ $G$ packs with $H(n, k)$ $\iff$ the complement, $\overline{G}$ contains $H(n, k)$. 
Applications

1. Scheduling, partitioning, and load balancing problems.

2. Deviation bounds for sums of random variables with limited dependence [Alon-Füredi, Janson-Ruciński, Pemmaraju].


A graph may have an equitable $k$-coloring but have no equitable $(k + 1)$-coloring.

An equitable 4-coloring of $K_{7,7}$.

Let $eq(G) = \min\{k : G \text{ has an equitable } m\text{-coloring for each } m \geq k\}$
To decide whether a graph has an equitable $k$-coloring is $NP$-complete even for $k = 3$.

This motivates extremal problems: if a graph $G$ is sparse, then it has low $eq(G)$.

“Sparse” may mean “Low maximum degree”, or “Low average degree”, or “Low degeneracy”, or a combination of those, or something else.
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Lo-o-o-ong proof.
**Theorem 1.** [Hajnal-Szemerédi] Every graph $G$ has $eq(G) \leq \Delta(G) + 1$.

**Conjecture 2.** [Chen-Lih-Wu] Let $G$ be a connected graph with maximum degree at most $r$. If $G$ is distinct from $K_{r+1}$, $K_{r,r}$ (for odd $r$), and is not an odd cycle, then $G$ has an equitable $r$-coloring.

The Chen-Lih-Wu Conjecture was proved: 1) For $r \leq 3$ [Chen-Lih-Wu], 2) For bipartite graphs [Lih-Wu], 3) For interval graphs [Chen-Lih-Yan], 4) For split graphs [Chen-Ko-Lih], 5) For outerplanar graphs [Yap-Zhang], 6) For planar graphs $G$ with $\Delta(G) \geq 13$ [Yap-Zhang], 7) For planar graphs $G$ with $\Delta(G) \geq 9$ [Nakprasit], 8) For graphs $G$ with $\text{avdeg}(G) \leq \Delta(G)/5$ [Kostochka-Nakprasit]
Ore-type problems

Ore’s Theorem. Every $n$-vertex graph $G$ with
\[ d(x) + d(y) \leq n - 2 \]
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Let the total edge degree of \( G \) be
\[ \theta(G') = \max\{d(x) + d(y) : xy \in E(G')\}. \]

\[ \delta(G) + \Delta(G) \leq \theta(G) \leq 2\Delta(G). \]

\[ \theta(G') = \Delta(L(G)) + 2. \]

\( \theta(G) \) equals the maximum degree of an edge of \( G \) in the total graph \( T(G) \).
Conjecture 3. [Kostochka-Yu] Every graph $G$ with $\theta(G) \leq 2r$ has $eq(G) \leq r + 1$.

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**Theorem 4.** [K-K] Every graph $G$ with $\theta(G) \leq 2r + 1$ has $eq(G) \leq r + 1$.

![Diagram](image)

$K_{m,2r-m}$ (odd $m$)
Conjecture 5. [K-K] If \( r \geq 3 \) and a connected graph \( G \) with \( \theta(G) \leq 2r \) differs from \( K_{r+1} \) and \( K_{m,2r-m} \) for all odd \( m \), then \( G \) has \( eq(G) \leq r \).

Theorem 6. [K-K] Conjecture 5 holds for \( r = 3 \).
Conjecture 5. [K-K] If $r \geq 3$ and a connected graph $G$ with $\theta(G) \leq 2r$ differs from $K_{r+1}$ and $K_{m,2r-m}$ for all odd $m$, then $G$ has $eq(G) \leq r$.


For odd $r \geq 3$, the Chen-Lih-Wu Conjecture does not describe disconnected graphs with max.deg $r$ that are not equitably $r$-colorable. For example, for an odd $r$, $K_{r,r} \cup K_{r,r}$ is equitably $r$-colorable, but $K_{r,r} \cup K_r$ is not.
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Observation 1: If $r$ is odd and $G$ is the disjoint union of $K_{r,r}$ and an $r$-equitable graph, then $G$ has no equitable $r$-coloring.
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**Observation 1:** If $r$ is odd and $G$ is the disjoint union of $K_{r,r}$ and an $r$-equitable graph, then $G$ has no equitable $r$-coloring.

**Observation 2:** If a spanning subgraph of $G$ is the disjoint union of $r$-equitable graphs, then $G$ is $r$-equitable.
A graph is \textit{r-equitable} if is \textit{r}-colorable and every its proper \textit{r}-coloring is equitable.

\textbf{Observation 1}: If \textit{r} is odd and \textit{G} is the disjoint union of \textit{K}_{r,r} and an \textit{r}-equitable graph, then \textit{G} has no equitable \textit{r}-coloring.

\textbf{Observation 2}: If a spanning subgraph of \textit{G} is the disjoint union of \textit{r}-equitable graphs, then \textit{G} is \textit{r}-equitable.

Clearly, a graph \textit{G} can be \textit{r}-equitable only for one \textit{r}. Call \textit{G} \textit{equitable} if it is \textit{r}-equitable for some \textit{r}.
Basic equitable graphs

$F_1$

$F_2$

$F_3$

$F_4$

$F_5$
More basic equitable graphs

$F_6$

$F_7$

$F_8$

$F_9$

$F_{10}$
Theorem [K-K] If \( r \geq 3 \) and a graph \( G \) with \( \theta(G) \leq 2r \) and \( |V(G)| \) divisible by \( r \) has an equitable \( r \)-coloring but has no nearly equitable \( r \)-coloring, then \( G \) is vertex-decomposable into \( r \)-basic graphs.

Conjecture 7. [K-K] If \( r \geq 3 \) is odd, then an \( r \)-colorable graph \( G \) with \( \Delta(G) \leq r \) does not have an equitable \( r \)-coloring if and only if a spanning subgraph of \( G \) is the disjoint union of \( K_{r,r} \) and basic \( r \)-equitable graphs.
Theorem [K-K] If $r \geq 3$ and a graph $G$ with $\theta(G) \leq 2r$ and $|V(G)|$ divisible by $r$ has an equitable $r$-coloring but has no nearly equitable $r$-coloring, then $G$ is vertex-decomposable into $r$-basic graphs.

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Theorem 8. [K-K] For every odd $r \geq 3$, if the Chen-Lih-Wu Conjecture holds for graphs $G$ with $\Delta(G) \leq r$, then Conjecture 7 holds for graphs $G$ with $\Delta(G) \leq r$. 
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Conjecture 10. [K-K] If $r \geq 3$, then an $r$-colorable graph $G$ with $\theta(G) \leq 2r$ does not have an equitable $r$-coloring if and only if a spanning subgraph of $G$ is the disjoint union of $K_{m,2r-m}$ for some odd $m$ and basic $r$-equitable graphs.
**Conjecture 10.** [K-K] If $r \geq 3$, then an $r$-colorable graph $G$ with $\theta(G) \leq 2r$ does not have an equitable $r$-coloring if and only if a spanning subgraph of $G$ is the disjoint union of $K_{m, 2r-m}$ for some odd $m$ and basic $r$-equitable graphs.

**Theorem 11.** [K-K] For every $r \geq 3$, if Conjecture 5 holds for graphs $G$ with $\theta(G) \leq r$, then Conjecture 10 holds for graphs $G$ with $\theta(G) \leq r$.

**Corollary 12.** [K-K] Conjecture 10 holds for $r = 3$. 