The Number of Spanning Trees in Graphs with a Given Degree Sequence

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Dedicated to Professor Paul Erdős on the occasion of his 80th birthday

ABSTRACT

Alon's [1] idea is slightly refined to prove that for each connected graph G with degree sequence $1 < k = d_1 \leq d_2 \leq \cdots \leq d_n$ the number $C(G)$ of spanning trees of G satisfies the inequality

$$d(G)k^{-n\,O(\log k/k)} \leq C(G) \leq d(G)/(n-1)\,,$$

where $d(G) = (\Pi_{i=1}^{n} d_i)$. An almost exact lower bound for $C(G)$ for 3-regular G on n vertices is also given.

1. INTRODUCTION

All graphs considered here are simple connected graphs. For a graph G on n vertices let $C(G)$ denote the number of spanning trees of G and $c(G) = (C(G))^{1/n}$. It is known (see [1, 2]) that for each k-regular graph G ($k > 1$),

$$c(G) < k\,. \qquad (1)$$

Alon [1] studied $c(k) = \lim_{n\to\infty} \inf\{c(G) \mid G \text{ is } k\text{-regular on } n \text{ vertices}\}$.

Theorem A [1]. *For each k-regular graph G,*

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$$c(G) \geq k^{1-O((\log\log k/\log k)^2)} .$$

Therefore, $c(k) \geq k^{1-O((\log\log k/\log k)^2)}$.

Theorem B [1]. *For each* $k > 2$, $\sqrt{2} \leq c(k) < ((k+1)^{k-2}(k-1))^{1/(k+1)} = k^{1-\Theta(1/k)}$. *In particular,* $2^{1/2} \leq c(3) \leq 2^{3/4}$.

The result of Theorem A together with (1) shows that the structure of a k-regular graph G unexpectedly does not affect too strongly the quantity $C(G)$. For example, if k is large enough, then we have $C(G) < C(H)$ for any k-regular G and $1.001k$-regular H on the same number of vertices. In this article we will follow the lines of Alon's proof to sharpen and generalize Theorem A. The main result is

Theorem 1. *Let G be a connected graph with degree sequence $1 < k = d_1 \leq d_2 \leq \cdots \leq d_n$. Denote $d(G) = (\Pi_{i=1}^{n} d_i)$. Then $d(G)k^{-n\,O(\log k/k)} \leq C(G) \leq d(G)/(n-1)$. Therefore,* $k^{1-O(\log k/k)} \leq c(k) \leq k^{1-\Theta(1/k)}$.

This means that the dependence of $C(G)$ on the structure of G for graphs with large minimal degree is rather weak. In [1] Alon asked about exact values of $c(k)$. We find here $c(3) = 2^{3/4}$ by proving

Theorem 2. *Let G be a graph whose vertex degrees are from $\{2, 3\}$, having m vertices of degree 3. Then*

$$C(G) \geq 2^{3(m+2)/4} , \tag{2}$$

provided $G \neq K_4$.

2. A LOWER BOUND FOR LARGE *k*

Let G be a (connected) graph with degree sequence $k = d_1 \leq d_2 \leq \cdots \leq d_n$ of vertices $v_1, v_2, \ldots, v_n$. Following Alon [1], for each $i \in \{1, \ldots, n\}$ choose, randomly (with a uniform distribution on the d_i vertices adjacent to v_i) and independently, a vertex $\Gamma(v_i)$ adjacent to v_i and orient the edge $(v_i, \Gamma(v_i))$ from v_i to $\Gamma(v_i)$. The number of components in the resulting oriented subgraph H of G is equal to the number of oriented cycles in H (oriented cycles of length 2 are possible). Note that the number of possible resulting oriented subgraphs H is $d(G)$. Every spanning tree T of G will be represented among these H exactly $n-1$ times (with cycles of length 2 using any one edge of T). Thus, $C(G) \leq d(G)/(n-1)$.

Lemma 1. *For each $i \in \{1, \ldots, n\}$ and any integer $t > 1$ the probability that v_i belongs to an oriented cycle in H of length t is at most $1/k$.*

Proof. Since $\Gamma(v_i)$ are chosen independently, we can consider the events consecutively. Let $w_0 = v_i$ and, for $j = 1, 2, \ldots, t-1$, $w_j = \Gamma(w_{j-1})$. If not all the vertices $w_0, w_1, \ldots, w_{t-1}$ are distinct or (w_0, w_{t-1}) is not an edge in G, then v_i

does not belong to an oriented cycle in H of length t. Otherwise, the probability that $v_i = \Gamma(w_{t-1})$ is $1/d(w_{t-1}) \leq 1/k$. □

Lemma 2. *For any integer $t > 1$ the expectation of the number of components of H with oriented cycles of length at most t is no more than $(n/k) \ln t$.*

Proof. By Lemma 1, the expected number of vertices on oriented cycles in H of length j is at most n/k. Hence the expected number of such cycles is at most n/kj. Thus, the desired expectation does not exceed $(n/k)(1/2 + 1/3 + \cdots + 1/t)$. □

To prove Theorem 1, it is enough now just to repeat the second half of Alon's proof [1] of Theorem A, keeping in mind Lemma 2. Thus, we only outline the arguments. Let $\mathscr{H}$ be the family of those oriented subgraphs H of G having at most $(2n/k) \ln k$ components of size k or less. Then each $H \in \mathscr{H}$ has less than $(3n/k) \ln k$ components. By Lemma 2, $|\mathscr{H}| \geq d(G)/2$. With each $H \in \mathscr{H}$ we associate a forest F_H by deleting an arbitrary edge from the unique cycle of each component. Then it can be seen that any such forest F can be obtained from at most $k^{(6n/k) \ln k}$ distinct $H \in \mathscr{H}$. Each F_H is contained in a spanning tree of G and any spanning tree of G contains at most

$$\sum_{i=0}^{(3n/k) \ln k} \binom{n-1}{i} = k^{n\, O(\log k/k)}$$

forests with less than $(3n/k) \ln k$ components. This completes the proof of Theorem 1.

3. ON GRAPHS WITH MAXIMAL DEGREE 3

Throughout the section for a graph H we will denote by $m(H)$ the number of vertices of degree 3 in H and $f(H) = 2^{3(m(H)+2)/4}$.

Let us count $C(G)$ for several G. Denote $H_1 = K_4$, $H_2 = K_4 \backslash e$, $H_3 = K_{3,3}$, H_4 – 3-prism; to obtain H_5, we subdivide an edge of K_4 by a vertex. It is an easy exercise to see that $C(H_1) = 16$, $C(H_2) = 8$, $C(H_3) = 81$, $C(H_4) = 75$, $C(H_5) = 24$. Thus, for $H_2 - H_5$, Theorem 2 is true.

Proof of Theorem 2. Suppose that $G = (V, E)$ is a minimum (on the number of edges) graph which is not K_4 such that $C(G) < f(G)$. Evidently, $|E| > 3$.

Claim 1. *G has no cut-edge.*

Proof. If G has a cut-edge, then for some s there is a path $P = (v_1, \ldots, v_s)$ such that $d(v_1) = d(v_s) = 3$, $d(v_2) = \cdots = d(v_{s-1}) = 2$ and all the edges of P are cut-edges. Let G_1 and G_2 be the components of the graph obtained from G by deleting the edges and the interior vertices of P. Both G_1 and G_2 have a vertex of degree 2 (namely, v_1 and v_s) and hence do not coincide with K_4. Note that $m(G_1) + m(G_2) = m(G) - 2$. By the minimality of G, $C(G_1)C(G_2) \geq f(G_1)f(G_2) = f(G)$. But $C(G) = C(G_1)C(G_2)$, a contradiction. □

Claim 2. *G is 3-regular.*

Proof. Suppose that G contains a vertex v with $N_G(v) = \{x, y\}$.

Case 1. $(x, y) \notin E$. Let $G_1 = (G \backslash v) \cup \{(x, y)\}$. If $G_1 = K_4$, then $G = H_5$, a contradiction. Otherwise by the minimality of $G, C(G_1) \geq f(G_1) = f(G)$. But $C(G) \geq C(G_1)$.

Case 2. $(x, y) \in E$. If $d(y) = 2$, then either $G = K_3$ and we are done, or the third edge incident with x is a cut-edge, which contradicts Claim 1. So, we can suppose that $N_G(x) = \{v, y, u\}$, $N_G(y) = \{v, x, w\}$. If $u = w$, then again either $G = K_4 \backslash e = H_2$ and we are done, or the third edge incident with w is a cut-edge. Thus, we can assume $u \neq w$. Let G_1 be obtained from G by contracting x, y, and v into a new vortex z (of degree 2). By the minimality of $G, C(G_1) \geq f(G_1) = 2^{-3/2}f(G)$. But each spanning tree of G_1 can be extended to a spanning tree of G by three ways [by adding any two edges of the triangle (x, v, y)]. Hence $C(G) \geq 3C(G_1) \geq 3f(G_1) > f(G)$. □

Claim 3. *G contains no subgraph $K_4 \backslash e$.*

Proof. Suppose that G contains a subgraph $K_4 \backslash e$ (see Fig. 1). Since G has no cut-edge, $w \neq v$. Consider $G_1 = G \backslash \{x, y, z, u\}$. By the minimality of G [$d(w) = 2$ in G_1], $C(G_1) \geq f(G_1) = 2^{-4.5}f(G)$. Because of $C(K_4 \backslash e) = 8$, each spanning tree T_1 of G_1 can be extended to a spanning tree T of G by:

(a) 8 ways such that T contains (x, w) and does not contain (u, v);
(b) 8 ways such that T contains (u, v) and does not contain (x, w);
(c) 8 ways such that T contains both (u, v) and (x, w).

Hence $C(G) \geq 24C(G_1) > f(G)$. □

Claim 4. *G contains no triangles.*

Proof. Suppose that G contains a triangle (x, y, z) and $\{(x, u), (y, v), (z, w)\} \subset E$. Due to Claim 3, the vertices u, v, and w are distinct. Let G_1 be obtained from G by contracting x, y, and z into a new vertex t. If $G_1 = K_4$, then $G = H_4$, a contradiction. Otherwise, by the minimality of G, $C(G_1) \geq f(G_1) = 2^{-3/2}f(G)$. But

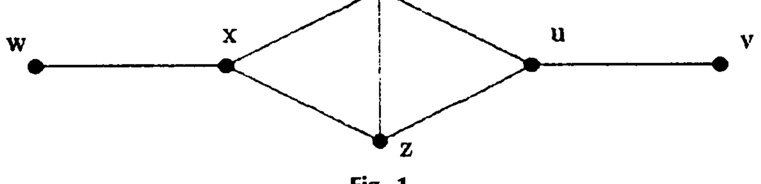

Fig. 1.

each spanning tree of G_1 can be extended to a spanning tree of G by three ways [by adding any two edges of the triangle (x, y, z)]. Hence $C(G) \geq 3C(G_1) \geq 3f(G_1) > f(G)$.

Now, consider the neighborhood of an arbitrary edge (u, v) of G [see Fig. 2(a)].

Consider $G_1 = (G \backslash \{v, u\}) \cup \{(x, y), (z, w)\}$ [see Fig. 2(b)]. If $G_1 = K_4$, then $G = H_3$, a contradiction. Otherwise, by the minimality of $G, C(G_1) \geq f(G_1) = 2^{-1.5}f(G)$. For each spanning tree T_1 of G_1, we will point out three spanning trees of G containing T_1 such that any spanning tree T of G will appear at most once.

Case 1. $E(T_1) \cap \{(x, y), (z, w)\} = \emptyset$. We extend T_1 by four ways adding an edge from $\{(x, u), (y, u)\}$ and an edge from $\{(z, v), (w, v)\}$.

Case 2. $(x, y) \in E(T_1)$, $(z, w) \not\in E(T_1)$. We add to $E(T_1)$ the edges (x, u), (y, u), and an edge incident to u (three ways).

Case 3. $(x, y) \not\in E(T_1)$, $(z, w) \in E(T_1)$. We do symmetrically to the Case 2.

Case 4. $E(T_1) \supset \{(x, y), (z, w)\}$. Then in $T_1 \backslash \{(x, y), (z, w)\}$ exactly one pair of elements of $\{x, y, z, w\}$ is connected by a path, and this pair is neither $\{x, y\}$ nor $\{z, w\}$. W.l.o.g. we suppose that this pair is $\{z, x\}$. Then we add to $E(T_1)$ the sets (a) $\{(x, u), (y, u), (z, v), (w, v)\}$, (b) $\{(v, u), (y, u), (z, v), (w, v)\}$, and (c) $\{(v, u), (y, u), (x, u), (w, v)\}$. □

To construct the examples for which equality in (2) holds, consider an arbitrary tree T with the maximal degree 3. Now, attach to every end-vertex v of T a copy of $K_4 \backslash e$ so that the degree of v becomes 3, and for some subset A of $E(T)$ replace each edge (v, w) by a copy of the graph on Figure 1 (with v and w as on Figure 1). Denote the resulting graph by $G(T, A)$. Recall that the number of end-vertices of T is equal to $m(T) + 2$. Since $C(K_4 \backslash e) = 8$, $m(G(T, A)) = 4(m(T) + 2 + |A|) - 2$ and each cut-edge belongs to any spanning tree, we have $C(G(T, A)) = 8^{m(T)+2+|A|} = 8^{(m(G(T,A))+2)/4}$. It can be proved that any graph with equality in (2) can be obtained from some $G(T, A)$ by replacing several paths by edges.

As to 3-regular graphs, consider a path $P_t = (v_1, \ldots, v_{2t})$. Attach to each end of P_t a copy of H_5 described above and replace for $i = 1, \ldots, t-1$ the edge (v_{2i}, v_{2i+1}) by a copy of $K_4 \backslash e$ so that the resulting graph $G(t)$ is 3-regular. We know that $C(H_5) = 24$, and $m(G(t)) = |V(G(t))| = 6 + 4t = 4(t + 2) - 2$. Hence, $C(G(t)) = 24^2 8^{t-1} = (9/8)\ 8^{(m(G(t))+2)/4} = (9/8)f(G(t))$. Thus, the bound here is also

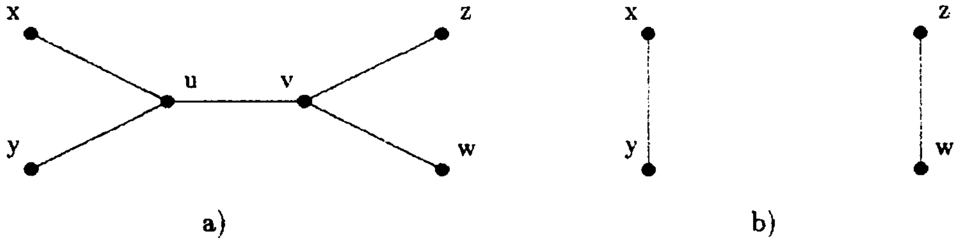

Fig. 2.

close to the truth. More careful consideration can show that $(9/8)f(G)$ is the lower bound of $C(G)$ for 3-regular graphs G except for K_4.

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