ON THE CORRÁDI-HAJNAL THEOREM AND A QUESTION OF DIRAC

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Abstract. In 1963, Corrádi and Hajnal proved that for all $k \geq 1$ and $n \geq 3k$, every graph $G$ on $n$ vertices with minimum degree $\delta(G) \geq 2k$ contains $k$ disjoint cycles. The bound $\delta(G) \geq 2k$ is sharp. Here we characterize those graphs with $\delta(G) \geq 2k - 1$ that contain $k$ disjoint cycles. This answers the simple-graph case of Dirac’s 1963 question on the characterization of $(2k - 1)$-connected graphs with no $k$ disjoint cycles.

Enomoto and Wang refined the Corrádi-Hajnal Theorem, proving the following Ore-type version: For all $k \geq 1$ and $n \geq 3k$, every graph $G$ on $n$ vertices contains $k$ disjoint cycles, provided that $d(x) + d(y) \geq 4k - 1$ for all distinct nonadjacent vertices $x, y$. We refine this further for $k \geq 3$ and $n \geq 3k + 1$: If $G$ is a graph on $n$ vertices such that $d(x) + d(y) \geq 4k - 3$ for all distinct nonadjacent vertices $x, y$, then $G$ has $k$ vertex-disjoint cycles if and only if the independence number $\alpha(G) \leq n - 2k$ and $G$ is not one of two small exceptions in the case $k = 3$. We also show how the case $k = 2$ follows from Lovász’ characterization of multigraphs with no two disjoint cycles.

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1. Introduction

For a graph $G = (V, E)$, let $|G| = |V|$, $|G| = |E|$, $\delta(G)$ be the minimum degree of $G$, and $\alpha(G)$ be the independence number of $G$. Let $\overline{G}$ denote the complement of $G$ and for disjoint graphs $G$ and $H$, let $G \vee H$ denote $G \cup H$ together with all edges from $V(G)$ to $V(H)$. The degree of a vertex $v$ in a graph $H$ is $d_H(v)$; when $H$ is clear, we write $d(v)$.

In 1963, Corrádi and Hajnal proved a conjecture of Erdős by showing the following:

Theorem 1.1 (6). Let $k \in \mathbb{Z}^+$. Every graph $G$ with (i) $|G| \geq 3k$ and (ii) $\delta(G) \geq 2k$ contains $k$ disjoint cycles.

Clearly, hypothesis (i) in the theorem is sharp. Hypothesis (ii) also is sharp. Indeed, if a graph $G$ has $k$ disjoint cycles, then $\alpha(G) \leq |G| - 2k$, since every cycle contains at least two vertices of $G - I$ for any independent set $I$. Thus $H := \overline{K}_{k+1} \vee K_{2k-1}$ satisfies (i) and has

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\[ \delta(H) = 2k - 1, \text{ but does not have } k \text{ disjoint cycles, because } \alpha(H) = k + 1 > |H| - 2k. \] 

There are several works refining Theorem 1.1. Dirac and Erdős \[8\] showed that if a graph \( G \) has many more vertices of degree at least 2 than vertices of degree at most \( 2k - 2 \), then \( G \) has \( k \) disjoint cycles. Dirac \[7\] asked:

**Question 1.2.** Which \((2k - 1)\)-connected graphs do not have \( k \) disjoint cycles?

He also resolved his question for \( k = 2 \) by describing all 3-connected multigraphs on at least 4 vertices in which every two cycles intersect. It turns out that the only simple 3-connected graphs with this property are wheels. Lovász \[22\] fully described all multigraphs in which every two cycles intersect.

The following result in this paper yields a full answer to Dirac’s question for simple graphs.

**Theorem 1.3.** Let \( k \geq 2 \). Every graph \( G \) with (i) \(|G| \geq 3k\) and (ii) \( \delta(G) \geq 2k - 1 \) contains \( k \) disjoint cycles if and only if

\[
\begin{align*}
(\text{H3}) & \quad \alpha(G) \leq |G| - 2k, \text{ and} \\
(\text{H4}) & \quad \text{if } k \text{ is odd and } |G| = 3k, \text{ then } G \neq 2K_k \cup \overline{K}_k \text{ and if } k = 2 \text{ then } G \text{ is not a wheel.}
\end{align*}
\]

Since for every independent set \( I \) in a graph \( G \) and every \( v \in I \), \( N(v) \subseteq V(G) - I \), if \( \delta(G) \geq 2k - 1 \) and \(|I| \geq |G| - 2k + 1 \), then \(|I| = |G| - 2k + 1 \) and \( N(v) = V(G) - I \) for every \( v \in I \). It follows that every graph \( G \) satisfying (ii) and not satisfying (H3) contains \( K_{2k-1, |G|-2k+1} \) and is contained in \( K_{|G|} - E(K_{|G|-2k+1}) \). The conditions of Theorem 1.3 can be tested in polynomial time.

Most likely, Dirac intended his question to refer to multigraphs; indeed, his result for \( k = 2 \) is for multigraphs. But the case of simple graphs is the most important in the question. In \[19\] we heavily use the results of this paper to obtain a characterization of \((2k - 1)\)-connected multigraphs that contain \( k \) disjoint cycles, answering Question 1.2 in full.

Studying Hamiltonian properties of graphs, Ore introduced the **minimum Ore-degree** \( \sigma_2 \): If \( G \) is a complete graph, then \( \sigma_2(G) = \infty \), otherwise \( \sigma_2(G) := \min\{d(x) + d(y) : xy \notin E(G)\} \).

Enomoto \[9\] and Wang \[24\] generalized the Corrádi-Hajnal Theorem in terms of \( \sigma_2 \):

**Theorem 1.4 (\[9, 24\]).** Let \( k \in \mathbb{Z}^+ \). Every graph \( G \) with (i) \(|G| \geq 3k\) and (ii) \( \sigma_2(G) \geq 4k - 1 \) contains \( k \) disjoint cycles.

Again \( H := \overline{K}_{k+1} \vee K_{2k-1} \) shows that hypothesis (E2) of Theorem 1.4 is sharp. What happens if we relax (E2) to (H2): \( \sigma_2(G) \geq 4k - 3 \), but again add hypothesis (H3)? Here are two interesting examples.

**Example 1.5.** Let \( k = 3 \) and \( Y_1 \) be the graph obtained by twice subdividing one of the edges \( wz \) of \( K_8 \), i.e., replacing \( wz \) by the path \( wxyz \). Then \(|Y_1| = 10 = 3k+1\), \( \sigma_2(Y_1) = 9 = 4k-3 \), and \( \alpha(Y_1) = 2 \leq |Y_1| - 2k \). However, \( Y_1 \) does not contain \( k = 3 \) disjoint cycles, since each cycle would need to contain three vertices of the original \( K_8 \) (see Figure 1.1(a)).

**Example 1.6.** Let \( k = 3 \). Let \( Q \) be obtained from \( K_{4,4} \) by replacing a vertex \( v \) and its incident edges \( uv, vx, vy, vz \) by new vertices \( u, u' \) and edges \( uu', uw, ux, u'y, u'z \); so \( d(u) = 3 = d(u') \) and contracting \( uu' \) in \( Q \) yields \( K_{4,4} \). Now set \( Y_2 := K_1 \vee Q \). Then \(|Y_2| = 10 = 3k+1\), \( \sigma_2(Y_2) = 9 = 4k-3 \), and \( \alpha(Y_2) = 4 \leq |Y_2| - 2k \). However, \( Y_2 \) does not contain \( k = 3 \) disjoint cycles, since each 3-cycle contains the only vertex of \( K_1 \) (see Figure 1.1(b)).
Our main result is:

**Theorem 1.7.** Let \( k \in \mathbb{Z}^+ \) with \( k \geq 3 \). Every graph \( G \) with

\[
(H1) \quad |G| \geq 3k + 1,
\]

\[
(H2) \quad \sigma_2(G) \geq 4k - 3, \quad \text{and}
\]

\[
(H3) \quad \alpha(G) \leq |G| - 2k
\]

contains \( k \) disjoint cycles, unless \( k = 3 \) and \( G \in \{Y_1, Y_2\} \). Furthermore, for fixed \( k \) there is a polynomial time algorithm that either produces \( k \) disjoint cycles or demonstrates that one of the hypotheses fails.

Theorem 1.7 is proved in Section 2. In Section 3 we discuss the case \( k = 2 \). In Section 4 we discuss connections to equitable colorings and derive Theorem 1.3 from Theorem 1.7 and known results.

Now we show examples demonstrating the sharpness of hypothesis (H2) that \( \sigma(G) \geq 4k - 3 \), then discuss some unsolved problems, and then review our notation.

**Example 1.8.** Let \( k \geq 3 \), \( Q = K_3 \) and \( G_k := \overline{K_{2k-2}} \lor (\overline{K_{2k-3}} + Q) \). Then \( |G_k| = 4k - 2 \geq 3k + 1 \), \( \delta(G_k) = 2k - 2 \) and \( \alpha(G_k) = |G_k| - 2k \). If \( G_k \) contained \( k \) disjoint cycles, then at least \( 4k - |G_k| = 2 \) would be 3-cycles; this is impossible, since any 3-cycle in \( G_k \) contains an edge of \( Q \). This construction can be extended. Let \( k = r + t \), where \( k + 3 \leq 2r \leq 2k \), \( Q' = K_{2t} \), and put \( H = G_r \lor Q' \). Then \( |H| = 4r - 2 + 2t = 2k + 2r - 2 \geq 3k + 1 \), \( \delta(H) = 2r - 2 + 2t = 2k - 2 \) and \( \alpha(H) = 2r - 2 = |H| - 2k \). If \( H \) contained \( k \) disjoint cycles, then at least \( 4k - |H| = 2t + 2 \) would be 3-cycles; this is impossible, since any 3-cycle in \( H \) contains an edge of \( Q \) or a vertex of \( Q' \).

There are several special examples for small \( k \). The constructions of \( Y_1 \) and \( Y_2 \) can be extended to \( k = 4 \) at the cost of lowering \( \sigma_2 \) to \( 4k - 4 \). Below is another small family of special examples. The blow-up of \( G \) by \( H \) is denoted by \( G[H] \); that is, \( V(G[H]) = V(G) \times V(H) \) and \( (x, y)(x', y') \in E(G[H]) \) if and only if \( xx' \in E(G) \), or \( x = x' \) and \( yy' \in E(H) \).

**Example 1.9.** For \( k = 4 \), \( G := C_5[K_3] \) satisfies \( |G| = 15 \geq 3k + 1 \), \( \delta(G) = 2k - 2 \) and \( \alpha(G) = 6 < |G| - 2k \). Since girth \( (G) = 4 \), we see that \( G \) has at most \( \lfloor |G|/4 \rfloor < k \) disjoint cycles. This example can be extended to \( k = 5, 6 \) as follows. Let \( I = \overline{K_{2k-8}} \) and \( H = G \lor I \). Then \( |G| = 2k + 7 \geq 3k + 1 \), \( \delta = 2k - 2 \) and \( \alpha(G) = 6 < |G| - 2k = 7 \). If \( H \) has \( k \) disjoint cycles then each of the at least \( k - (2k - 8) = 8 - k \) cycles that do not meet \( I \) use 4 vertices of \( G \), and the other cycles use at least 2 vertices of \( G \). Then \( 15 = |G| \geq 2k + 2(8 - k) = 16 \), a contradiction.
Unsolved problems. 1. For every fixed $k$, we know only a finite number of extremal examples. It would be very interesting to describe all graphs $G$ with $\sigma_2(G) = 4k - 4$ that do not have $k$ disjoint cycles, but this most likely would need new techniques and approaches.

2. Recently, there were several results in the spirit of the Corrâdi-Hajnal Theorem giving degree conditions on a graph $G$ sufficient for the existence in $G$ of $k$ disjoint copies of such subgraphs as chorded cycles \cite[4]{1} and $\Theta$-graphs \cite[5]{3}. It could be that our techniques can help in similar problems.

3. One also may try to sharpen the above-mentioned theorem of Dirac and Erdős \cite[8]{3}.

Notation. A bud is a vertex with degree 0 or 1. A vertex is high if it has degree at least $2k - 1$, and low otherwise. For vertex subsets $A, B$ of a graph $G = (V, E)$, let

$$\|A, B\| := \sum_{u \in A} |\{uv \in E(G) : v \in B\}|.$$

Note $A$ and $B$ need not be disjoint. For example, $\|V, V\| = 2\|G\| = 2|E|$. We will abuse this notation to a certain extent. If $A$ is a subgraph of $G$, we write $\|A, B\|$ for $\|V(A), B\|$, and if $A$ is a set of disjoint subgraphs, we write $\|A, B\|$ for $\bigcup_{H \in A} \|V(H), B\|$. Similarly, for $u \in V(G)$, we write $\|u, B\|$ for $\|\{u\}, B\|$. Formally, an edge $e = uv$ is the set $\{u, v\}$; we often write $\|e, A\|$ for $\|\{u, v\}, A\|$.

If $T$ is a tree or a directed cycle and $u, v \in V(T)$ we write $uTv$ for the unique subpath of $T$ with endpoints $u$ and $v$. We also extend this: if $w \notin T$, but has exactly one neighbor $u \in T$, we write $wTv$ for $w(T + w + uw)v$. Finally, if $w$ has exactly two neighbors $u, v \in T$, we may write $wTw$ for the cycle $wuTv$.

2. Proof of Theorem \cite[1.7]{1,1}

Suppose $G = (V, E)$ is an edge-maximal counterexample to Theorem \cite[1.7]{1}. That is, for some $k \geq 3$, (H1)--(H3) hold, and $G$ does not contain $k$ disjoint cycles, but adding any edge $e \in E(G)$ to $G$ results in a graph with $k$ disjoint cycles. The edge $e$ will be in precisely one of these cycles, so $G$ contains $k - 1$ disjoint cycles, and at least three additional vertices. Choose a set $C$ of disjoint cycles in $G$ so that:

- (O1) $|C|$ is maximized;
- (O2) subject to (O1) $\sum_{C \in C} |C|$ is minimized;
- (O3) subject to (O1) and (O2) the length of a longest path $P$ in $R := G \cup C$ is maximized;
- (O4) subject to (O1), (O2), and (O3) $\|R\|$ is maximized.

Call such a $C$ an optimal set. We prove in Subsection 2.1 that $R$ is a path, and in Subsection 2.2 that $|R| = 3$. We develop the structure of $C$ in Subsection 2.3. Finally, in Subsection 2.4 these results are used to prove Theorem \cite[1.7]{1}.

Our arguments will have the following form. We will make a series of claims about our optimal set $C$, and then show that if any part of a claim fails, then we could have improved $C$ by replacing a sequence $C_1, \ldots, C_t \in C$ of at most three cycles by another sequence of cycles $C_1', \ldots, C_t'$. Naturally, this modification may also change $R$ or $P$. We will express the contradiction by writing “$C_1', \ldots, C_t', R', P'$ beats $C_1, \ldots, C_t, R, P$, ” and may drop $R'$ and $R$ or $P'$ and $P$ if they are not involved in the optimality criteria.

This proof implies a polynomial time algorithm. We start by adding enough extra edges—at most $3k$—to obtain from $G$ a graph with a set $C$ of $k$ disjoint cycles. Then we remove the
extra edges in $C$ one at a time. After removing an extra edge, we calculate a new collection $C'$. This is accomplished by checking the series of claims, each in polynomial time. If a claim fails, we calculate a better collection (again in polynomial time) and restart the check, or discover an independent set of size greater than $|G| - 2k$. As there can be at most $n^4$ improvements, corresponding to adjusting the four parameters $[O1]−[O4]$ this process ends in polynomial time.

We now make some simple observations. Recall that $|C| = k − 1$ and $R$ is acyclic. By $[O2]$ and our initial remarks, $|R| ≥ 3$. Let $a_1$ and $a_2$ be the endpoints of $P$. (Possibly, $R$ is an independent set, and $a_1 = a_2$.)

Claim 2.1. For all $w ∈ V(R)$ and $C ∈ C$, if $|w, C| ≥ 2$ then $3 ≤ |C| ≤ 6 − |w, C|$. In particular, (a) $|w, C| ≤ 3$, (b) if $|w, C| = 3$ then $|C| = 3$, and (c) if $|C| = 4$ then the two neighbors of $w$ in $C$ are nonadjacent.

Proof. Let $\overrightarrow{C}$ be a cyclic orientation of $C$. For distinct $u, v ∈ N(w) ∩ C$, the cycles $wu\overrightarrow{C}vw$ and $wu\overrightarrow{C}vw$ have length at least $|C|$ by $[O2]$. Thus $2 |C| ≤ |wwu\overrightarrow{C}vw| + |vwu\overrightarrow{C}vw| = |C| + 4$, so $|C| ≤ 4$. Similarly, if $|w, C| ≥ 3$ then $3 |C| ≤ |C| + 6$, and so $|C| ≤ 3$. 

The next claim is a simple corollary of condition $[O2]$.

Claim 2.2. If $xy ∈ E(R)$ and $C ∈ C$ with $|C| ≥ 4$ then $N(x) ∩ N(y) ∩ C = ∅$.

2.1. $R$ is a path. Suppose $R$ is not a path. Let $L$ be the set of buds in $R$; then $|L| ≥ 3$.

Claim 2.3. For all $C ∈ C$, distinct $x, y, z ∈ V(C), i ∈ [2]$, and $u ∈ V(R − P)$:

(a) $\{ux, uy, a_z\} ∉ E$;
(b) $|\{u, a_i\}, C| ≤ 4$;
(c) $\{a_i, x, i_a, y, a_{3−i}, a_x, x\} ∉ E$;
(d) if $|\{a_i, a_2\}, C| ≥ 5$ then $|u, C| = 0$;
(e) $|\{a_i, a_2\}, R| ≥ 1$; in particular $|a_i, R| = 1$ and $|P| ≥ 2$;
(f) $4 − |u, R| ≤ |\{u, a_i\}, C|$ and $|\{u, a_i\}, D| = 4$ for at least $|C| − |u, R|$ cycles $D ∈ C$.

Proof. (a) Else $ux(C − z)yu, Pa_z$ beats $C, P$ by $[O3]$ (see Figure 2.1(a)).
(b) Else $|C| = 3$ by Claim 2.1. Then there are distinct $p, q, r ∈ V(C)$ with $up, uq, a_i r ∈ E$, contradicting (a).
(c) Else $a_i x(C − z)ya_i, (P − a_i) a_{3−i}, z u$ beats $C, P$ by $[O3]$ (see Figure 2.1(b)).
(d) Suppose $|\{a_i, a_2\}, C| ≥ 5$ and $p ∈ N(u) ∩ C$. By Claim 2.1 $|C| = 3$. Pick $j ∈ [2]$ with $pa_j ∈ E$, preferring $|a_j, C| = 2$. Then $V(C) − p ⊆ N(a_{3−j})$, contradicting (c).
(e) Since $a_i$ is an end of the maximal path $P$, we get $N(a_i) ∩ R ⊆ P$; so $a_i u ∉ E$. By (b) (2.1)

$$4(k − 1) ≥ |\{u, a_i\}, V \backslash R| ≥ 4k − 3 − |\{u, a_i\}, R|.$$  

Thus $|\{u, a_i\}, R| ≥ 1$. Hence $G[R]$ has an edge, $|P| ≥ 2$, and $|a_i, P| = |a_i, R| = 1$.
(f) By (2.1) and (c), $|\{u, a_i\}, V \backslash R| ≥ 4|C| − |u, R|$. Using (f), this implies the second assertion, and $|\{u, a_i\}, C| + 4(|C| − 1) ≥ 4|C| − |u, R|$ implies the first assertion. 

Claim 2.4. $|P| ≥ 3$. In particular, $a_1 a_2 ∉ E(G)$.

Proof. Suppose $|P| ≤ 2$. Then $|u, R| ≤ 1$. As $|L| ≥ 3$, there is a bud $c ∈ L \setminus \{a_1, a_2\}$. By Claim 2.3(f), there exists $C = z \ldots z_i a_i ∈ C$ such that $|\{c, a_i\}, C| = 4$ and $|\{c, a_1\}, C| ≥ 3$. 

If $\|c, C\| = 3$ then the edge between $a_1$ and $C$ contradicts Claim 2.3. If $\|c, C\| = 1$ then $\|\{a_1, a_2\}, C\| = 5$, contradicting Claim 2.3. Therefore, we assume $\|c, C\| = 2 = \|a_1, C\|$ and $\|a_2, C\| \geq 1$. By Claim 2.3, $N(a_1) \cup N(a_2) = N(c)$, so there exists $z_i \in N(a_1) \cap N(a_2)$ and $z_j \in N(c) - z_i$. Then $a_1 a_2 z_1$ beats $C$, $P$ by (O3).

Claim 2.5. Let $c \in L - a_1 - a_2$, $C \in C$, and $i \in [2]$.

(a) $\|a_1, C\| = 3$ if and only if $\|c, C\| = 0$, and if and only if $\|a_2, C\| = 3$.
(b) There is at most one cycle $D \in C$ with $\|a_i, D\| = 3$.
(c) For every $C \in C$, $\|a_i, C\| \geq 1$ and $\|c, C\| \leq 2$.
(d) If $\|a_i, C\| = 4$ then $\|a_i, C\| = 2 = \|c, C\|$.

Proof. (a) If $\|c, C\| = 0$ then by Claims 2.1 and 2.3, $\|a_i, C\| = 3$. If $\|a_i, C\| \geq 3$ then by Claim 2.3, $\|c, C\| \leq 1$. By Claim 2.3, $\|a_3, C\| \geq 2$, and by Claim 2.3, $\|c, C\| = 0$.
(b) As $c \in L$, $\|c, R\| \leq 1$. Thus Claim 2.3 implies $\|c, D\| = 0$ for at most one cycle $D \in C$.
(c) Suppose $\|c, C\| = 3$. By Claim 2.3, $\|\{a_1, a_2\}, C\| = 0$. By Claims 2.4 and 2.3:
$$4k - 3 \leq \|\{a_1, a_2\}, R \cup C \cup (V - R - C)\| \leq 2 + 0 + 4(k - 2) = 4k - 6,$$
a contradiction. Thus $\|c, C\| \leq 2$. Thus by Claim 2.3, $\|a_i, C\| \geq 1$.
(d) Now (d) follows from (a) and (c).

Claim 2.6. $R$ has no isolated vertices.

Proof. Suppose $c \in L$ is isolated. Fix $C \in C$. By Claim 2.3, $\|\{c, a_1\}, C\| = 4$. By Claim 2.3, $\|a_1, C\| = 2 = \|c, C\|$; so $d(c) = 2(k - 1)$. By Claim 2.3, $N(a_1) \cap C = N(c) \cap C$. Let $w \in V(C) - N(c)$. Then $d(w) \geq 4k - 3 - d(c) = 2k - 1 = 2|C| + 1$. Therefore, either $\|w, R\| \geq 1$ or $|N(w) \cap D| = 3$ for some $D \in C$. In the first case, $c(C - w)c$ beats $C$ by (O4). In the second case, by Claim 2.3 there exists some $x \in N(a_1) \cap D$. Then $c(C - w)c, w(D - x)w$ beats $C, D$ by (O3).

Claim 2.7. $L$ is an independent set.

Proof. Suppose $c_1 c_2 \in E(L)$. By Claim 2.4, $c_1, c_2 \notin P$. By Claim 2.3 and using $k \geq 3$, there is $C \in C$ with $\|\{c_1, c_2\}, C\| = 4$ and $\|\{a_1, c_2\}, C\|, \|\{a_2, c_1\}, C\| \geq 3$. By Claim 2.3, $\|a_1, C\| = 2 = \|c_1, C\|$; so $\|a_2, C\|, \|c_2, C\| \geq 1$. By Claim 2.3, $N(a_1) \cap C, N(a_2) \cap C \subseteq N(c_1) \cap C$. Then there are distinct $x, y \in N(c_1) \cap C$ with $x a_1, x a_2, y a_1 \in E$. If $x c_2 \in E$ then $c_1 c_2 x a_1, y a_1 P a_2$ beats $C, P$ by (O3). Else $a_1 P a_2 x a_1, c_1(C - x)c_2 c_1$ beats $C, P$ by (O1).
Claim 2.8. If $|L| \geq 3$ then for some $D \in \mathcal{C}$, $\|l, C\| = 2$ for every $C \in \mathcal{C} - D$ and every $l \in L$.

**Proof.** Suppose some $D_1, D_2 \in \mathcal{C}$ and $l_1, l_2 \in L$ satisfy $D_1 \neq D_2$ and $\|l_1, D_1\| \neq 2 \neq \|l_2, D_2\|$. CASE 1: $l_j \notin \{a_1, a_2\}$ for some $j \in [2]$. Say $j = 1$. For $i \in [2]$: $\{\{a_i, l_1\}, D_1\| \neq 4$ by Claim 2.5(a); $\{\{a_i, l_1\}, D_2\| = 4$ by Claim 2.3(l); $\|a_i, D_2\| = 2$ by Claim 2.5(d). Then $l_2 \notin \{a_1, a_2\}$. By Claim 2.7, $l_1 l_2 \notin E(G)$. Claim 2.5(c) yields the contradiction:

$$4k - 3 \leq \|\{l_1, l_2\}, R \cup D_1 \cup D_2 \cup (V - R - D_1 - D_2)\| \leq 2 + 3 + 3 + 4(k - 3) = 4k - 4.$$

CASE 2: $\{l_1, l_2\} \subseteq \{a_1, a_2\}$. Let $c \in L - l_1 - l_2$. As above, $\|\{l_1, c\}, D_1\| \neq 4$, and so $\|c, D_2\| = 2 = \|l_1, D_2\|$. This implies $l_1 \neq l_2$. By Claim 2.5(a), $\|l_2, D_2\| = 1$. Thus $\|l_2, c\|, D_1\| = 4$; so $\|c, D_1\| = 2$, and $\|l_1, D_1\| = 1$. With Claim 2.4 this yields the contradiction:

$$4k - 3 \leq \|\{l_1, l_2\}, R \cup D_1 \cup D_2 \cup (V - R - D_1 - D_2)\| \leq 2 + 3 + 3 + 4(k - 3) = 4k - 4.$$  

$\square$

Claim 2.9. $R$ is a subdivided star (possibly a path).

**Proof.** Suppose not. Then we claim $R$ has distinct leaves $c_1, d_1, c_2, d_2 \in L$ such that $c_1 R d_1$ and $c_2 R d_2$ are disjoint paths. Indeed, if $R$ is disconnected then each component has two distinct leaves by Claim 2.6. Else $R$ is a tree. As $R$ is not a subdivided star, it has distinct vertices $s_1$ and $s_2$ with degree at least three. Deleting the edges and interior vertices of $s_1 R s_2$ yields disjoint trees containing all leaves of $R$. Let $T_i$ be the tree containing $s_i$, and pick $c_i, d_i \in T_i$.

By Claim 2.8 using $k \geq 3$, there is a cycle $C \in \mathcal{C}$ such that $\|l, C\| = 2$ for all $l \in L$. By Claim 2.3(a), $N(a_1) \cap C = N(l) \cap C = N(a_2) \cap C = \{w_1, w_3\}$ for $l \in L - a_1 - a_2$. Then replacing $C$ in $C$ with $w_1 c_1 R d_1 w_1$ and $w_3 c_2 R d_2 w_3$ yields $k$ disjoint cycles. $\square$

Claim 2.10. $R$ is a path or a star.

![Figure 2.2](image-url) **Figure 2.2. Claim 2.10**

**Proof.** By Claim 2.9, $R$ is a subdivided star. If $R$ is neither a path nor a star then there are vertices $r, p, d$ with $\|r, R\| \geq 3$, $\|p, R\| = 2$, $d \in L - a_1 - a_2$ and (say) $pa_1 \in E$. Then $a_2 R d$ is disjoint from $pa_1$ (see Figure 2.2(a)). By Claim 2.5(c), $d(d) \leq 1 + 2(k - 1) = 2k - 1$. Then:

$$\|p, V - R\| \geq 4k - 3 - \|p, R\| - d(d) \geq 4k - 5 - (2k - 1) = 2k - 4 \geq 2.$$

In each of the following cases, $R \cup C$ has two disjoint cycles, contradicting (O1).

CASE 1: $\|p, C\| = 3$ for some $C \in \mathcal{C}$. Then $|C| = 3$. By Claim 2.5(a), if $\|d, C\| = 0$ then $\|a_1, C\| = 3 = \|a_2, C\|$. Then for $w \in C$, $wa_1 pw$ and $a_2 (C - w) a_2$ are disjoint cycles (see Figure 2.2(b)). Else by Claim 2.5(c), $\|d, C\|, \|a_2, C\| \in \{1, 2\}$. By Claim 2.3(l),
\[\|\{d, a_2\}, C\| \geq 3, \text{ so there are } l_1, l_2 \in \{a_2, d\} \text{ with } \|l_1, C\| \geq 1 \text{ and } \|l_2, C\| = 2; \text{ say } w \in N(l_1) \cap C. \text{ If } l_2w \in E \text{ then } w_lR_l w \text{ and } p(C - w)p \text{ are disjoint cycles (see Figure 2.2(c)); else } l_1wpR_l \text{ and } l_2(C - w)l_2 \text{ are disjoint cycles (see Figure 2.2(d)).}

\text{CASE 2: There are distinct } C_1, C_2 \in \mathcal{C} \text{ with } \|p, C_1\|, \|p, C_2\| \geq 1. \text{ By Claim 2.8 for some } i \in [2] \text{ and all } c \in L, \|c, C_i\| = 2. \text{ Let } w \in N(p) \cap C_i. \text{ If } wa_1 \in E \text{ then } D := wpz_1w \text{ is a cycle and } G[(C_i - w) \cup a_2Rd] \text{ contains cycle disjoint from } D. \text{ Else, if } w \in N(c), \text{ then } a_1(C_i - w)a_1 \text{ and } cwpRc \text{ are disjoint cycles. Else, by Claim 2.1 there exist vertices } u \in N(a_2) \cap N(d) \cap C_i \text{ and } v \in N(a_1) \cap C_1 - u. \text{ Then } u_2Rdu \text{ and } a_1v(C_i - u)wpa_1 \text{ are disjoint cycles.}

\text{CASE 3: Otherwise. Then using (2.2), } \|p, V - R\| = 2 = \|p, C\| \text{ for some } C \in \mathcal{C}. \text{ In this case, } k = 3 \text{ and } d(p) = 4. \text{ By (H2), } d(a_2), d(d) \geq 5. \text{ Say } C = \{C, D\}. \text{ By Claim 2.8(b), } \|\{a_2, d\}, D\| \leq 4. \text{ Thus,}

\[\|\{a_2, d\}, C\| = \|\{a_2, d\}, (V - R - D)\| \geq 10 - 2 - 4 = 4.\]

\text{By Claim 2.5(c), } \|a_2, C\| = \|d, C\| = 2 \text{ and } \|a_1, C\| \geq 1. \text{ Say } w \in N(a_1) \cap C. \text{ If } wp \in E \text{ then } dR_{a_2}(C - w)d \text{ contains a cycle disjoint from } wa_1pw. \text{ Else, by Claim 2.3(a) there exists } x \in N(a_2) \cap N(d) \cap C. \text{ If } x \neq w \text{ then } x_2Rdx \text{ and } wa_1p(C - x)w \text{ are disjoint cycles. Else } x = w, \text{ and } x_2Rdx \text{ and } p(C - w)p \text{ are disjoint cycles.}

\text{Lemma 2.11. } R \text{ is a path.}

\text{Proof. Suppose } R \text{ is not a path. Then it is a star with root } r \text{ and at least three leaves, any of which can play the role of } a_i \text{ or a leaf in } L - a_1 - a_2. \text{ Thus Claim 2.5(c) implies } \|l, C\| \in \{1, 2\} \text{ for all } l \in L \text{ and } C \in \mathcal{C}. \text{ By Claim 2.8 there is } D \in \mathcal{C} \text{ such that for all } l \in L \text{ and } C \in \mathcal{C} - D, \|l, C\| = 2. \text{ By Claim 2.3(a) there is } l \in L \text{ such that for all } c \in L - l, \|c, D\| = 2. \text{ Fix distinct leaves } l', l'' \in L - l.

\text{Let } Z = N(l') - R \text{ and } A = V \setminus (Z \cup \{r\}). \text{ By the first paragraph, every } C \in \mathcal{C} \text{ satisfies } |Z \cap C| = 2, \text{ so } |A| = |G| - 2k + 1. \text{ For a contradiction, we show that } A \text{ is independent.}

\text{Note } A \cap R = L, \text{ so by Claim 2.7 } A \cap R \text{ is independent. By Claim 2.3(a),}

\[ (2.3) \quad \text{for all } c \in L \text{ and for all } C \in \mathcal{C}, N(c) \cap C \subseteq Z.\]

\text{Therefore, } \|L, A\| = 0. \text{ By Claim 2.1(c), for all } C \in \mathcal{C}, C \cap A \text{ is independent. Suppose, for a contradiction, } A \text{ is not independent. Then there exist distinct } C_1, C_2 \in \mathcal{C}, v_1 \in A \cap C_1, \text{ and } v_2 \in A \cap C_2 \text{ with } v_1v_2 \in E. \text{ Subject to this choose } C_2 \text{ with } \|v_1, C_2\| \text{ maximum. Let } Z \cap C_1 = \{x_1, x_2\} \text{ and } Z \cap C_2 = \{y_1, y_2\}.

\text{CASE 1: } \|v_1, C_2\| \geq 2. \text{ Choose } i \in [2] \text{ so that } \|v_1, C_2 - y_i\| \geq 2. \text{ Then define } C_1^* := v_1(C_2 - y_i)v_1, C_2^* := l'x_1(C_1 - v_1)x_2l', \text{ and } P^* := y_1l''rl \text{ (see Figure 2.3(a)). By (2.3), } P^* \text{ is a path and } C_2^* \text{ is a cycle. Then } C_1^*, C_2^*, P^* \text{ beats } C_1, C_2, P \text{ by (O3)}.

\text{CASE 2: } \|v_1, C_2\| < 1. \text{ Then for all } C \in \mathcal{C}, \|v_1, C\| \leq 2 \text{ and } \|v_1, C_2\| = 1; \text{ so } \|v_1, C\| = \|v_1, C_2 \cup (C - C_2)\| \leq 1 + 2(k - 2) = 2k - 3. \text{ By (2.3), } \|v_1, L\| = 0 \text{ and } d(l) \leq 2k - 1. \text{ By (H2), } \|v_1, r\| = \|v_1, L\| = (4k - 3) - \|v_1, C\| - d(l) \leq (4k - 3) - (2k - 3) - (2k - 1) = 1, \text{ and } v_1r \in E. \text{ Let } C_1^* := l'x_1(C_1 - v_1)x_2l', C_2^* := l'y_1(C_2 - v_2)y_2l'', \text{ and } P^* := v_2r_1l \text{ (see Figure 2.3(b)). Then } C_1^*, C_2^*, P^* \text{ beats } C_1, C_2, P \text{ by (O3)}. \text{ } \square

2.2. } |R| = 3. \text{ By Lemma 2.11, } R \text{ is a path, and by Claim 2.1 } |R| \geq 3. \text{ Next we prove } |R| = 3. \text{ First, we prove a claim that will also be useful in later sections.}
Claim 2.12. Let $C$ be a cycle, $P = v_1 v_2 \ldots v_s$ be a path in $R$, and $1 < i < s$. At most one of the following two statements holds.

1. (a) $|x, v_i P v_{i-1}| \geq 1$ for all $x \in C$ or (b) $|x, v_1 P v_{i-1}| \geq 2$ for two $x \in C$;
2. (c) $|y, v_i P v_s| \geq 2$ for some $y \in C$ or (d) $N(v_i) \cap C \neq \emptyset$ and $|v_i P v_s, C| \geq 2$.

Proof. Suppose (1) and (2) hold. If (c) holds then the disjoint graphs $G[v_i P v_s + y]$ and $G[v_i P v_{i-1} \cup C - y]$ contain cycles. Else (d) holds, but (c) fails; say $z \in N(v_i) \cap C$ and $z \notin N(v_{i+1} P v_s)$. If (a) holds then $G[v_1 P v_i + z]$ and $G[v_{i+1} P v_s \cup C - z]$ contain cycles. If (b) holds then $G[v_1 P v_{i-1} + w]$ and $G[v_i P v_s \cup C - w]$ contain cycles, where $|w, v_1 P v_{i-1}| \geq 2$. □

Suppose, for a contradiction, $|R| \geq 4$. Say $R = a_1 a_1' a_1'' \ldots a_2 a_2' a_2''$. It is possible that $a''_1 \in \{a''_2, a''_3\}$, etc. Set $e_i := a_i a_i' = \{a_i, a_i'\}$ and $F := e_1 \cup e_2$.

Claim 2.13. If $C \in C$, $h \in [2]$ and $|e_h, C| \geq |e_{3-h}, C|$ then $|C, F| \leq 7$; if $|C, F| = 7$ then $|C| = 3$, $|a_h, C| = 2$, $|a_h', C| = 3$, $|a_h'' Ra_{3-h}, C| = 2$, and $N(a_h) \cap C = N(e_{3-h}) \cap C$.

Proof. We will repeatedly use Claim 2.12 to obtain a contradiction to (1) by showing that $G[C \cup R]$ contains two disjoint cycles. Suppose $|C, F| \geq 7$ and say $h = 1$. Then $|e_1, C| \geq 4$.

There is $x \in e_1$ with $|x, C| \geq 2$. Thus $|C| \leq 4$ by Claim 2.1 and if $|C| = 4$ then no vertex in $C$ has two adjacent neighbors in $F$. Then (1) holds with $v_1 = a_1$ and $v_i = a_i'$, even when $|C| = 4$.

If $|e_1, C| = 4$, as is the case when $|C| = 4$, then $|e_2, C| \geq 3$. If $|C| = 4$ there is a cycle $D := y a'_2 a''_2 y$ for some $y, z \in C$. As (a) holds, $G[a_i R a'_2 \cup C - y - z]$ contains another disjoint cycle. Thus, $|C| = 3$. As (c) must fail with $v_i = a'_2$, (a) and (c) hold for $v_i = a_i'$ and $v_1 = a_2$, a contradiction. Then $|e_1, C| \geq 5$. If $|a_1, C| = 3$ then (a) and (c) hold with $v_1 = a_1$ and $v_i = a_i'$. Now $|a_1, C| = 2$, $|a'_1, C| = 3$ and $|a''_1 Ra_2, C| \geq 2$. If there is $b \in P - e_1$ and $c \in N(b) \cap V(C) \setminus N(a_1)$ then $G[a'_1 R a_2 + c]$ and $G[a_1 (C - c) a_1]$ both contain cycles. For every $b \in R - e_1$, $N(b) \cap C \subseteq N(a_1)$. If $|a''_1 Ra_2, C| \geq 3$, (c) holds for $v_1 = a_1$ and $v_i = a''_2$, contradicting that (1) holds. Now $|a''_1 Ra_2, C| = |e_1, C| = 2$ and $N(a_1) = N(e_2)$. □

Lemma 2.14. $|R| = 3$ and $m := \max\{|C| : C \in C\} = 4$.

Proof. Let $t = \{|C \in C : |F, C| \leq 6\}$ and $r = \{|C \in C : |C| \geq 5\}$. It suffices to show $r = 0$ and $|R| = 3$: then $m = 4$, and $|V(C)| = |G| - |R| \geq 3(k - 1) + 1$ implies some $C \in C$ has
length 4. Choose $R$ so that:

(P1) $R$ has as few low vertices as possible, and subject to this,

(P2) $R$ has a low end if possible.

Let $C \in \mathcal{C}$. By Claim 2.13 $\|F, C\| \leq 7$. By Claim 2.1 if $|C| \geq 5$ then $\|a, C\| \leq 1$ for all $a \in F$; so $\|F, C\| \leq 4$. Thus $r \leq t$. Hence

\[(2.4) \quad 2(4k - 3) \leq \|F, (V \setminus R) \cup R\| \leq 7(k - 1) - t - 2r + 6 \leq 7k - t - 2r - 1.\]

Therefore, $5 - k \geq t + 2r \geq 3r$, so $r = 0$ and $t \leq 2$, with $t = 2$ only if $k = 3$.

**CASE 1:** $k - t \geq 3$. That is, there exist distinct cycles $C_1, C_2 \in \mathcal{C}$ with $\|F, C_i\| \geq 7$. In this case, $t \leq 1$: if $k = 3$ then $\mathcal{C} = \{C_1, C_2\}$ and $t = 0$; if $k > 3$ then $t \leq 2$. For both $i \in [2]$, Claim 2.13 yields $\|F, C_i\| = 7$, $|C_i| = 3$, and there is $x_i \in V(C_i)$ with $\|x_i, R\| = 1$ and $\|y, R\| = 3$ for both $y \in V(C_i - x_i)$. Moreover, there is a unique index $j = \beta(i) \in [2]$ with $\|a_j', C_i\| = 3$. For $j \in [2]$, put $I_j := \{i \in [2] : \beta(i) = j\}$; that is, $I_j = \{i \in [2] : \|a_j', C_i\| = 3\}$. Then $V(C_i) - x_i = N(a_{\beta(i)}) \cap C_i = N(e_{\beta(i)}) \cap C_i$. As $x_i, a_{\beta(i)} \notin E$, one of $x_i, a_{\beta(i)}$ is high. As we can switch $x_i$ and $a_{\beta(i)}$ (by replacing $C_i$ with $a_{\beta(i)}(C_i - x_i)a_{\beta(i)}$ and $R$ with $R - a_{\beta(i)} + x_i$), we may assume $a_{\beta(i)}$ is high.

Suppose $I_j \neq \emptyset$ for both $j \in [2]$: say $\|a_1', C_1\| = \|a_2', C_2\| = 3$. Then for all $B \in \mathcal{C}$ and $j \in [2]$, $a_j$ is high, and either $\|a_j, B\| \leq 2$ or $\|F, B\| \leq 6$. Since $t \leq 1$, we get

$2k - 1 \leq d(a_j) = \|a_j, B \cup F\| + \|a_j, C - B\| \leq \|a_j, B\| + 2(2k - 2) + t \leq 2k - 2 + \|a_j, B\|.$

Thus $N(a_j) \cap B \neq \emptyset$ for all $B \in \mathcal{C}$. Let $y_j \in N(a_{\beta(j)}) \cap C_j$. Then using Claim 2.13 $y_j \in N(a_j)$, and $a_1'(C_1 - y_1)a_1', a_2'(C_2 - y_2)a_2', a_1y_1a_2y_2a_1$ beats $C_1, C_2$ by (O1).

Otherwise, say $I_1 = \emptyset$. If $B \in \mathcal{C}$ with $\|F, B\| \leq 6$ then $\|e_1, B\| + 2\|a_2, B\| \leq \|F, B\| + \|a_2, B\| \leq 9$. Thus, using Claim 2.13

\[2(4k - 3) \leq d(a_1) + d(a_1') + 2d(a_2) = 5 + \|e_1, C\| + 2\|a_2, C\| \leq 5 + 6(k - 1 - t) + 9t \]

$\Rightarrow 2k \leq 5 + 3t.$

Since $k - t \geq 3$ (by the case), we see $3(k - t) + (5 + 3t) \geq 3(3) + 2k$ and so $k \geq 4$. Since $t \leq 1$, in fact $k = 4$ and $t = 1$, and equality holds throughout: say $B$ is the unique cycle in $\mathcal{C}$ with $\|F, B\| \leq 6$. Then $\|a_2, B\| = \|e_1, B\| = 3$. Using Claim 2.13 $d(a_1) + d(a_1') = \|e_1, R\| + \|e_1, C - B\| + \|e_1, B\| = 3 + 4 + 3 = 10$, and $d(a_1), d(a_2) \geq (4k - 3) - d(a_2) = 13 - (1 + 4 + 3) = 5$, so $d(a_1) = d(a_2) = 5$. Note $a_1$ and $a_2$ share no neighbors: they share none in $R$ because $R$ is a path, they share none in $C - B$ by Claim 2.13 and they share no neighbor $b \in B$ lest $a_1a_1'ba_1$ and $a_2(B - b)a_2$ beat $B$ by (O1). Thus every vertex in $V_e - e_1$ is high.

Since $\|e_1, B\| = 3$, first suppose $\|a_2, B\| \geq 2$, say $B - b \subseteq N(a_1)$. Then $a_1(B - b)a_1, a_1'a_1'b$ beat $B$, $R$ by (P1) (see Figure 2.4(a)). Now suppose $\|a_2, B\| \geq 2$, this time with $B - b \subseteq N(a_2')$. Since $d(a_2) = 5$ and $\|a_2', R \cup B\| \leq 2$, there exists $c \in C = \mathcal{C} - B$ with $a_1c \in E(G)$. Now $c \in N(a_2)$ by Claim 2.13 so $a_2'(B - b)a_1, a_2'(C - c)a_2'$, and $a_1a_2b$ beat $B, C$, and $R$ by (P1) (see Figure 2.4(b)).

**CASE 2:** $k - t \leq 2$. That is, $\|F, C\| \leq 6$ for all but at most one $C \in \mathcal{C}$. Then, since $5 - k \geq t$, we get $k = 3$ and $\|F, V\| \leq 19$. Say $\mathcal{C} = \{C, D\}$, so $\|F, C \cup D\| \geq 2(4k - 3) - \|F, R\| = 2(4 \cdot 3 - 3) - 6 = 12$. By Claim 2.13 $\|F, C\|, \|F, D\| \geq 5$. If $|R| \geq 5$, then for the (at most two) low vertices in $R$, we can choose distinct vertices in $R$ not adjacent to them. Then $\|R, V - R\| \geq 5|R| - 2 - \|R, R\| = 3|R|$. Thus we may assume $\|R, C\| \geq \lfloor 3|R|/2 \rfloor \geq |R| + 3 \geq 8$. Let $w' \in C$ be such that $q =$
Suppose $\parallel w, R \parallel = \max \{\parallel w, R \parallel : w \in C \}$. Let $N(w') \cap R = \{v_{i_1}, \ldots, v_{i_q}\}$ with $i_1 < \ldots < i_q$. Suppose $q \geq 4$. If $\parallel v_1Rv_{i_2}, C - w' \parallel \geq 2$ or $\parallel v_{i_2+1}Rv_{s}, C - w' \parallel \geq 2$, then $G[C \cup R]$ has two disjoint cycles. Otherwise, $\parallel R, C - w' \parallel \leq 2$, contradicting $\parallel R, C \parallel \geq \parallel R \parallel + 3$. Similarly, if $q = 3$, then $\parallel v_1Rv_{i_2-1}, C - w' \parallel \leq 1$ and $\parallel v_{i_2+1}Rv_{s}, C - w' \parallel \leq 1$ yielding $\parallel v_{i_2}, C \parallel = \parallel R, C \parallel \geq (\parallel R \parallel + 3) - 2 - (3 - 1) \geq 4$, a contradiction to Claim 2.1. Therefore, $q \leq 2$, and hence $\parallel R \parallel + 3 \leq \parallel R, C \parallel \leq 2\parallel C \parallel$. It follows that $\parallel R \parallel = 5$, $\parallel C \parallel = 4$ and $\parallel w, R \parallel = 2$ for each $w \in C$. This in turn yields that $G[C \cup R]$ has no triangles and $\parallel v_i, C \parallel \leq 2$ for each $i \in [5]$. By Claim 2.1.3, $\parallel F, C \parallel \leq 6$, so $\parallel v_3, C \parallel = 2$. Thus we may assume that for some $w \in C$, $N(w) \cap R = \{v_1, v_3\}$. Then $\parallel e_2, C \parallel = \parallel e_2, C - w \parallel \leq 1$, lest there exist a cycle disjoint from $wv_1v_2v_3w$ in $G[C \cup R]$. Therefore, $\parallel e_1, C \parallel \geq 8 - 1 - 2 = 5$, a contradiction to Claim 2.1.3. This yields $\parallel R \parallel \leq 4$.

Claim 2.15. Either $a_1$ or $a_2$ is low.

Proof. Suppose $a_1$ and $a_2$ are high. Then since $\parallel R, V \parallel \leq 19$, we may assume $a_1'$ is low. Suppose there is $c \in C$ with $ca_2 \in E$ and $\parallel a_1, C\parallel - c \parallel \geq 2$. If $a_2'c \in E$, then $R \cup C$ contains two disjoint cycles; so $a_1'c \notin E$ and hence $c$ is high. Thus either $a_1(C - c)a_1$ is shorter than $C$ or the pair $a_1(C - c)a_1, ca_2a_2'a_1$ beats $C, R$ by (P2). Thus if $ca_2 \in E$ then $\parallel a_1, C\parallel - c \parallel \leq 1$. As $a_2$ is high, $\parallel a_2, C \parallel \geq 1$ and hence $\parallel a_1, C \parallel = \parallel a_1, C \setminus N(a_2) \parallel + \parallel a_1, N(a_2) \parallel \leq 2$. Similarly, $\parallel a_1, D \parallel \leq 2$. Since $a_1$ is high, we see $\parallel a_1, C \parallel = \parallel a_1, D \parallel = 2$, and $d(a_1) = 5$. Hence

$$\parallel a_2 \parallel \cap C \subseteq N(a_1) \cap C \quad \text{and} \quad \parallel a_2 \parallel \cap D \subseteq N(a_1) \cap D.$$  

As $a_2$ is high, $d(a_2) = 5$ and in (2.5) equalities hold. Also $d(a_1') = 4 \leq d(a_2')$.

If there are $c \in C$ and $i \in [2]$ with $ca_i, ca'_i \in E$ then by $[\text{2.1.2}]$, $\parallel C \parallel = 3$. Also $ca'_i, a_2'$, $a_2'c_3, a_2c_3 - i(C - c)$ beats $C, R$ by either (P1) or (P2). (Recall $N(a_1) \cap C = N(a_2) \cap C$ and neighbors of $a_2$ in $C$ are high.) Then $N(a_i) \cap N(a'_i) = \emptyset$. Thus the set $N(a_i) - R = N(a_2) - R$ contains no low vertices. Also, if $\parallel a'_i, C \parallel \geq 1$ then $\parallel C \parallel = 3$: else $C$ has the form $c_1a_2c_3c_4c_1$, where $a_1c_1, a_1c_3 \in E$, and so $a_1c_1a_2c_3c_4a_2c_3$ beats $C, R$ by either (P1) or (P2). Thus $\parallel C \parallel = 3$ and $a'_i, c_i \in E$ for some $c \in V(C) - N(a_1)$. If $\parallel a'_2, C \parallel \geq 1$, we have disjoint cycles $ca'_2, a_1(C - c)a_1$ and $D$. Then $\parallel a'_2, C \parallel = 0$, so $d(a'_1) \leq 2 + |D \setminus N(a_1)| \leq 4$. Now $a'_1$ and $a'_2$ are symmetric, and we have proved that $\parallel a'_2, C \parallel \geq \parallel a'_1, C \parallel \geq 4$. Similarly, $\parallel a'_1, D \parallel \geq \parallel a'_2, D \parallel \geq 1$, a contradiction to $d(a'_1), d(a'_2) \geq 4$. \hfill \Box

By Claim 2.1.3, we can choose notation so that $a_1$ is low.
Claim 2.16. If $a'_1$ is low then each $v \in V \setminus e_1$ is high.

Proof. Suppose $v \in V - e_1$ is low. Since $a_1$ is low, all vertices in $R - e_1$ are high, so $v \in C$ for some $C \in \mathcal{C}$. Then $C' := ve_1v$ is a cycle and so by (O2) $|C| = 3$. Since $a_2$ is high, $|a_2, C| \geq 1$. As $v$ is low, $va_2 \notin E$. Since $a'_1$ is low, it is adjacent to the low vertex $v$, and $|a'_1, C - v| \leq 1$. Then $C', a'_2a_2(C - v)$ beats $C, R$ by (P1).

Claim 2.17. If $|C| = 3$ and $|e_1, C|, |e_2, C| \geq 3$, then either
(a) $|c, e_1| = 1 = |c, e_2|$ for all $c \in V(C)$ or
(b) $a'_1$ is high and there is $c \in V(C)$ with $|c, R| = 4$ and $C - c$ has a low vertex.

Proof. If (a) fails then $|c, e_1| = 2$ for some $i \in [2]$ and $c \in C$. If $|e_{3-i}, C - c| \geq 2$ then there is a cycle $C'' \subseteq C \cup e_{3-i} - c$, and $R \cup C$ contains disjoint cycles $ce_i c$ and $C''$. Else,

$$|c, R| = |c, e_1| + (|C, e_{3-i}| - |C - c, e_{3-i}|) \geq 2 + (3 - 1) = 4 = |R|.$$ 

If $C - c$ has no low vertices then $ce_1 c, e_2(C - c)$ beats $C, R$ by (P1). Then $C - c$ contains a low vertex $c'$. If $a'_1$ is low then $c'a'_1 c'$ and $ca_2a'_2 c$ are disjoint cycles. Thus, (b) holds.

CASE 2.1: $|D| = 4$. By (O2) $G[R \cup D]$ does not contain a 3-cycle. Then $5 \leq d(a_2) \leq 3 + |a_2, C| \leq 6$. Thus $d(a_1), d(a'_1) \geq 3$.

Suppose $|e_1, D| \geq 3$. Pick $v \in N(a_1) \cap D$ with minimum degree, and $v' \in N(a'_1) \cap D$. Since $N(a_1) \cap D$ and $N(a'_1) \cap D$ are nonempty, disjoint and independent, we see $vv' \in E$. Say $D = K_{2,2}$ and low vertices are adjacent, $D' := a_1a'_1 v' v a_1$ is a 4-cycle and $v$ is the only possible low vertex in $D$. Note $a_1w \notin E$: else $a_1 w w' v a_1$ beats $D, R$ by (P1). As $|e_1, D| \geq 3$, $a'_1 w' \in E$. Also note $|e_2, w w'| = 0$: else $G[a_2, a'_2, w, w']$ contains a 4-path $R'$, and $D', R'$ beats $D, R$ by (P1). Similarly, replacing $D'$ by $D'' := a_1a'_1 w' v a_1$ yields $|e_2, v' v'| = 0$. Then $|e_1 \cup e_2, D| \leq 3 + 1 = 4$, a contradiction. Thus

$$(2.6) \quad |e_1, D| \leq 2 \quad \text{and so} \quad |R, D| \leq 6.$$ 

Suppose $d(a'_1) = 3$. Then $|a'_1, D| \leq 1$. Then there is $uv \in E(D)$ with $|a'_1, uv| = 0$. Thus $d(u), d(v), d(a_2) \geq 6$, and $|a_2, C| = 3$. Now $|C| = 3$, $|G'| = 11$, and there is $w \in N(u) \cap N(v)$. If $w \in C$ put $C' := a_2(C - w) a_2$; else $C' = C$. In both cases, $|C'| = |C|$ and $|w w w w| = 3 < |D|$, so $C'$, $w w w w$ beats $C, D$ by (O2). Thus $d(a'_1) \geq 4$. If $d(a_1) = 3$ then $d(a_2), d(a'_2) \geq 9 - 3 = 6$, and $|a_2, C| \geq 3$. By (2.6),

$$|R, C| \geq 3 + 4 + 6 + 6 - |R, R| - |R, D| \geq 19 - 6 - 6 = 7,$$

contradicting Claim 2.13. Then $d(a_1) = 4 \leq d(a'_1)$ and by (2.6), $|e_1, C| \geq 3$. Thus (2.6) fails for $C$ in place of $D$; so $|C| = 3$. As $|a_2, C| \geq 2$ and $|a'_2, C| \geq 1$, Claim 2.17 implies either (a) or (b) of Claim 2.17 holds. If (a) holds then (a) and (d) of Claim 2.12 both hold, and so $G[C \cup R]$ has two disjoint cycles. Else, Claim 2.17 gives $a'_1$ is high and there is $c \in C$ with $|c, R| = 4$. As $a'_1$ is high, $|R, C| \geq 7$. Now $|c, R| = 4$ contradicts Lemma 2.13.

CASE 2.2: $|C| = |D| = 3$ and $|R, V| = 18$. Then $d(a_1) + d(a'_2) = 9 = d(a'_1) + d(a_2)$, $a_1$ and $a'_1$ are low, and by Claim 2.16 all other vertices are high. Moreover, $d(a'_1) \leq d(a_1)$, since

$$18 = |R, V| = d(a'_1) - d(a_1) + 2d(a_1) + d(a'_2) + d(a_2) \geq d(a'_1) - d(a_1) + 9 + 9.$$ 

Suppose $d(a'_1) = 2$. Then $d(v) \geq 7$ for all $v \in V - a_1a'_1 a_2$. In particular, $C \cup D \subseteq N(a_2)$. If $d(a_1) = 2$ then $d(a'_2) \geq 7$, and $G = Y_1$. Else $|a_1, C \cup D| \geq 2$. If there is $c \in C$ with $V(C) - c \subseteq N(a_1)$, then $a_1(C - c) a_1, a'_1 a'_2 a_2 c$ beats $C, R$ by (P1). Else $d(a_1) = 3, d(a'_2) = 6$, and there are $c \in C$ and $d \in D$ with $c, d \in N(a_1)$. If $ca_2 \in E$ then $C \cup R$ contains disjoint
cycles $a_1c'a_1'a_1$ and $a_2(C - c)a_2$, so assume not. Similarly, assume $da'_2 \notin E$. Since $d(d) \geq 7$ and $a_1', a_2' \notin N(d)$, we see $cd \in E(G)$. Then there are three disjoint cycles $a'_2(C - c)a'_2$, $a_2(D - d)a_2$, and $a_1cda_1$. Thus $d(a'_1) \geq 3$.

Suppose $d(a'_1) = 3$. Say $a'_1v \in E$ for some $v \in D$. As $d(a_2) \geq 6$, $\|a_2, D\| \geq 2$. Then $e_2 + D - v$ contains a 4-path $R'$. Thus $a_1v \notin E$: else $ve_1v, R'$ beats $D, R$ by (P1). Also $\|a_1, D - v\| \leq 1$: else $a_1(D - v)a_1, va'_1a'_2a_2$ beats $D, R$ by (P1). Then $\|a_1, D\| \leq 1$.

Suppose $\|a_1, C\| \geq 2$. Pick $c \in C$ with $C - c \subseteq N(a_1)$. Then
\begin{equation}
(2.7)
\end{equation}\
\[a_2c \notin E:\]

else $a_1(C - c)a_1, a'_1a'_2a_2$ beats $C, R$ by (P1). Then $\|a_2, C\| = 2$ and $\|a_2, D\| = 3$. Also $a_1c \notin E$: else picking a different $c$ violates (2.7). As $a'_1 \notin E$, $\|c, D\| = 3$ and $a'_2c \in E(G)$. Then $a_1(C - c)a_1, a_2(D - v)a_2$ and $va'_2a'_2c$ are disjoint cycles. Otherwise, $\|a_1, C\| \leq 1$ and $d(a_1) \leq 3$. Then $d(a_1) = 3$ since $d(a_1) \geq d(a'_1)$.

Now $d(a'_2) = 6$. Say $D = vbb'v$ and $a_1b \in E$. As $b'a'_1 \notin E$, $d(b') \geq 9 - 3 = 6$. Since $\|e_2, V\| = 12$, we see that $a_2$ and $a'_2$ have three common neighbors. If one is $b'$ then $D' := a_1a'_1vba_1, b'e_2b'$, and $C$ are disjoint cycles; else $\|b', C\| = 3$ and there is $c' \in C$ with $\|c', e_2\| = 2$. Then $D', c'e_2c'$ and $b'(C - c')b'$ are disjoint cycles. Thus, $d(a'_1) = 4$.

Since $a_1$ is low and $d(a_1) \geq d(a'_1)$, we see $d(a_1) = d(a'_1) = 4$ and $\{|a_1, a'_1\}, C \cup D\| = 5$, so we may assume $\|e_1, C\| \geq 3$. If $\|e_2, C\| \geq 3$, then because $a'_2$ is low, Claim (2.17(a)) holds. Now, $V(C) \subseteq N(e_1)$ and there is $x \in e_1 = xy$ with $\|x, C\| \geq 2$. First suppose $\|x, C\| = 3$. As $x$ is low, $x = a_1$. Pick $c \in N(a_2) \cap C$, which exists because $\|a_2, C \cup D\| \geq 4$. Then $a_1(C - c)a_1, a'_1a'_2a_2$ beats $C, R$ by (P1). Now suppose $\|x, C\| = 2$. Let $c \in C \cap N(x)$. Then $x(C - c)x, yce_2$ beats $C, R$ by (P1).

CASE 2.3: $|C| = |D| = 3$ and $\|R, V\| = 19$. Say $\|C, R\| = 7$ and $\|D, R\| = 6$.

CASE 2.3.1: $a'_1$ is low. Then $\|a'_1, C \cup D\| \leq 4 - \|a'_1, R\| = 2$, so by Claim (2.13) $\|e_2, C\| = 5$ with $\|a_2, C\| = 2$. Then $5 \leq d(a_2) \leq 6$.

If $d(a_2) = 5$ then $d(a_1) = d(a'_1) = 4$ and $d(a'_2) = 6$. Then $\|a_2, D\| = 2$ and $\|a'_2, D\| = 1$. Say $D = b_1b_2b_3b_1$, where $a_2b_2, a_3b_3 \in E$. As $a'_1$ is low, (a) of Claim (2.17) holds. Then $\|b_1, a_1a'_1a'_2\| = 2$, and there is a cycle $D' \subseteq G[b_1a_1a'_1a'_2]$. Then $a_2(D - b_1)a_2$ and $D'$ are disjoint.

If $d(a_2) = 6$ then $\|a_2, D\| = 3$. Let $c_1 \in C - N(a_2)$. By Claim (2.13) $\|c_1, R\| = 1$, so $c_1$ is high, and $\|c_1, D\| \geq 2$. If $\|a'_2, D\| \geq 1$, then (a) and (d) hold in Claim (2.12) for $v_1 = a_2$ and $v_2 = a'_2$, so $G[D \cup a_1a'_2a_2]$ has two disjoint cycles, and $c_2e_1c_3c_2$ contains a third. Therefore, assume $\|a'_2, D\| = 0$, and so $d(a'_2) = 5$. Thus $d(a_1) = d(a'_1) = 4$. Again, $\|e_1, D\| = 3 = \|a_2, D\|$. Now there are $x \in e_1$ and $b \in V(D)$ with $D - b \not\subseteq N(x)$. As $a'_1$ is low and has two neighbors in $R$, if $\|x, D\| = 3$ then $x = a_1$. Anyway, using Claim (2.17) $G[R + b - x]$ contains a 4-path $R'$, and $x(D - b)x, R'$ beats $R$ by (P1).

CASE 2.3.2: $a'_1$ is high. Since $19 = \|R, V\| \geq d(a_1) + d(a'_1) + 2(9 - d(a_1)) \geq 23 - d(a_1)$, we get $d(a_1) = 4$ and $d(a'_1) = d(a'_2) = d(a_2) = 5$. Choose notation so that $C = c_1c_2c_3c_1$, $D = b_1b_2b_3b_1$, and $\|c_1, R\| = 1$. By Claim (2.13) there is $i \in [2]$ with $\|a_i, C\| = 2, \|a'_i, C\| = 3$, and $a_ic_i \notin E$. If $i = 1$ then every low vertex is in $N(a_1) - a'_1 \subseteq D \cup C'$, where $C' = a_1c_2c_3a_1$. Then $C', c_1a'_1a'_2a_2$ beats $C, R$ by (P1). Thus let $i = 2$. Now $\|a_2, C\| = 3 = \|a_2, D\|$. Say $a_2b_2, a_3b_3 \in E$. Also $\|a'_2, D\| = 0$ and $\|e_1, D\| = 4$. Then $\|b_2, e_1\| = 2$ for some $j \in [3]$. If $j = 1$ then $b_1c_1b_1$ and $a_2b_3b_2a_2$ are disjoint cycles. Else, say $j = 2$. By inspection, all low vertices are contained in $\{a_1, b_1, b_3\}$. If $b_1$ and $b_3$ are high then $b_1e_1b_2, b_1b_2e_2$ beats $D, R$ by (P1). Else there is a 3-cycle $D' \subseteq G[D + a_1]$ that contains every low vertex of $G$. Pick $D'$
with $b_1 \in D'$ if possible. If $b_2 \notin D'$ then $D'$ and $b_2a_1'a_2'a_2b_2'$ are disjoint cycles. If $b_3 \notin D'$ then $D'$, $b_2a_2'a_2'a_2b_3$ beats $D, R$ by (P1). Else $b_1 \notin D', a_1b_1 \notin E$, and $b_1$ is high. If $b_1a_1' \in E$ then $D', b_1'a_2'a_2$ beats $D, R$ by (P1). Else, $\|b_1, C\| = 3$. Then $D', b_1c_2b_2$, and $c_3c_2c_3$ are disjoint cycles. □

2.3. Key Lemma. Now $|R| = 3$; say $R = a_1a'2$. By Lemma 2.14 the maximum length of a cycle in $C$ is 4. Fix $C = w_1 \ldots w_4w_1 \in C$.

Lemma 2.18. If $D \in C$ with $\|R, D\| \geq 7$ then $|D| = 3$, $\|R, D\| = 7$ and $G[R \cup D] = K_6 - E(K_3)$.

Proof. Since $\|R, D\| \geq 7$, there exists $a \in R$ with $\|a, D\| \geq 3$. By Claim 2.14 $|D| = 3$. If $\|a_i, D\| = 3$ for any $i \in [2]$, then (a) and (c) in Claim 2.12 hold, violating [O1]. Then $\|a_1, D\| = \|a_2, D\| = 2$ and $\|a'2, D\| = 3$. If $G[R \cup D] \neq K_6 - K_3$ then $N(a_1) \cap D \neq N(a_2) \cap D$. Then there is $w \in N(a_1) \cap N(a_2)$ with $\|a_i, D - w\| = 2$. Then $w_1a'2w$ and $a_2(D - w)a_2$ are disjoint cycles. □

Lemma 2.19. Let $D \in C$ with $D = z_1\ldots z_1z_1$. If $\|C, D\| \geq 8$ then $\|C, D\| = 8$ and

$$W := G[C \cup D] \in \{K_{4,4}, K_1 \vee K_{3,3}, \overline{K}_3 \vee (K_1 + K_3)\}.$$  

Proof. First suppose $|D| = 4$. Suppose

\[ (2.8) \]

Then $C' := C - C - D + T + C'$ is an optimal choice of $k - 1$ disjoint cycles, since $C$ is optimal. By Lemma 2.14 $|C'| \leq 4$. Thus $C'$ beats $C$ by [O2].

CASE 1: $\Delta(W) = 6$. By symmetry, assume $d_W(w_4) = 6$. Then $\{|z_i, z_{i+1}|, C - w_4| \geq 2 \text{ for some } i \in \{1, 3\}$). Then $\ref{2.8}$ holds with $T = w_4z_{4-i}z_{i-4}w_4$.

CASE 2: $\Delta(W) = 5$. Say $z_1, z_2, z_3 \in N(w_4)$. Then $\{|z_i, z_4|, C - w_1| \geq 2 \text{ for some } i \in \{1, 3\}$). Then $\ref{2.8}$ holds with $T = w_1z_{1+i}z_{3-i}w_1$.

CASE 3: $\Delta(W) = 4$. Then $W$ is regular. If $W$ has a triangle then $\ref{2.8}$ holds. Else, say $w_1z_1, w_1z_3 \in E$. Then $z_1, z_3 \notin N(w_2) \cup N(w_4)$, so $z_2, z_4 \in N(w_2) \cup N(w_4)$, and $z_1, z_3 \in N(w_3)$.

Now, suppose $|D| = 3$.

CASE 1: $d_W(z_i) = 6$ for some $h \in [3]$. Say $h = 3$. If $w_i, w_{i+1} \in N(z_j)$ for some $i \in [4]$ and $j \in [2]$, then $z_3w_{i+2}w_{i+3}z_3, z_3w_{i+2}z_{j-1}z_j$ beats $C, D$ by [O2]. Else for all $j \in [2], \|z_j, C\| = 2$, and the neighbors of $z_j$ in $C$ are nonadjacent. If $w_i \in N(z_1) \cap N(z_2) \cap C$, then $z_3w_{i+1}w_{i+2}z_3, z_3w_{i+1}z_3$ are preferable to $C, D$ by [O2]. Wence $W = K_1 \vee K_{3,3}$.

CASE 2: $d_W(z_h) \leq 5$ for every $h \in [3]$. Say $d(z_1) = 5 = d(z_2), d(z_3) = 4$, and $w_1, w_2, w_3 \in N(z_1)$. If $N(z_1) \cap C \neq N(z_2) \cap C$ then $W - z_3$ contains two disjoint cycles, preferable to $C, D$ by [O2]. If $w_i \in N(z_3)$ for some $i \in \{1, 3\}$ then $W - w_4$ contains two disjoint cycles. Then $N(z_3) = \{w_2, w_4\}$, and so $W = \overline{K}_3 \vee (K_1 + K_3)$, where $V(K_1) = \{w_4\}, w_2z_1z_2w_2 = K_3$, and $V(K_3) = \{w_1, w_3, z_3\}$. □

Claim 2.20. For $D \in C$, if $\{|w_1, w_3\}, D\| \geq 5$ then $\|C, D\| \leq 6$. If also $|D| = 3$ then $\{|w_2, w_4\}, D\| = 0$.

Proof. Assume not. Let $D = z_1\ldots z_1z_1$. Then $\{|w_1, w_3\}, D\| \geq 5$ and $\|C, D\| \geq 7$. Say $\|w_1, D\| \geq \|w_3, D\|, \{z_1, z_2, z_3\} \subseteq N(w_1)$, and $z_i \in N(w_3)$.
Suppose \(\|w_1, D\| = 4\). Then \(|D| = 4\). If \(\|z_h, C\| \geq 3\) for some \(h \in \{4\}\) then there is a cycle \(B \subseteq G[w_2, w_3, w_4, z_h]\); so \(B, w_1z_{h+1}z_{h+2}w_1\) beats \(C, D\) by (O2). Else there are \(j \in \{l-1, l+1\}\) and \(i \in \{2, 3, 4\}\) with \(z_jw_j \in E\). Then \(z_lz_j[w_iw_3]z_l, w_1(D - z_l - z_j)w_1\) beats \(C, D\) by (O2) where \([w_iw_3] = w_3\) if \(i = 3\).

Else, \(\|w_1, D\| = 3\) by assumption, there is \(i \in \{2, 4\}\) with \(\|w_i, D\| \geq 1\). If \(|D| = 3\), applying Claim 2.12 with \(P := w_1w_iw_3\) and cycle \(D\) yields two disjoint cycles in \((D \cup C) - w_{6-i}\), contradicting (O2). Therefore, suppose \(|D| = 4\). Because \(w_1z_1z_2w_1\) and \(w_1z_2z_3w_1\) are triangles, there do not exist cycles in \(G[w_i, w_3, z_1, z_i]\) or \(G[w_i, w_3, z_1, z_4]\) by (O2). Then \(\|\{w_1, w_3\}, \{z_3, z_4\}\|, \|\{w_i, w_3\}, \{z_1, z_4\}\| \leq 1\). Since \(\|\{w_i, w_3\}, D\| \geq 3\), one has a neighbor in \(z_2\). If both are adjacent to \(z_2\), then \(w_1z_2z_3w_1\) and \(w_1z_1z_2z_3w_1\) beat \(C, D\) by (O2). Then \(\|\{w_1, w_3\}, z_2\| = 1 = \|\{w_i, w_3\}, z_1\| = \|\{w_i, w_3\}, z_3\|\). Let \(z_m\) be the neighbor of \(w_i\). Then \(w_1w_1z_mw_i; w_3(D - z_m)w_3\) beat \(C, D\) by (O2).

Suppose \(|D| = 3\) and \(\|\{w_1, w_3\}, D\| \geq 5\). If \(\|\{w_2, w_4\}, D\| \geq 1\), then \(C \cup D\) contains two triangles, and these are preferable to \(C\) by (O2).

For \(v \in N(C)\), set type\((v) = i \in [2]\) if \(N(v) \cap C \subseteq \{w_1, w_{i+2}\}\). Call \(v\) light if \(\|v, C\| = 1\); else \(v\) is heavy. For \(D = z_1 \ldots z_kz_1 \in C\), put \(H := H(D) := G[R \cup D]\).

Claim 2.21. If \(\|\{a_1, a_2\}, D\| \geq 5\) then there exists \(i \in [2]\) such that

(a) \(\|C, H\| \leq 12\) and \(\|w_i, w_{i+2}\), \(H\| \leq 4\);
(b) \(\|C, H\| = 12\);
(c) \(N(w_1) \cap H = N(w_{i+2}) \cap H = \{a_1, a_2\}\) and \(N(w_3) \cap H = N(w_{i-3}) \cap H = V(D) \cup \{a'\}\).

Proof. By Claim 2.1, \(|D| = 3\). Choose notation so that \(\|a_1, D\| = 3\) and \(z_2, z_3 \in N(a_2)\).

Using that \(\{w_1, w_3\}\) and \(\{w_2, w_4\}\) are independent and Lemma 2.19

(2.9) \(\|C, H\| = \|C, V - (V - H)\| \geq 2(4k - 3) - 8(k - 2) = 10\).

Let \(v \in V(H)\). As \(K_4 \subseteq H\), \(H - v\) contains a 3-cycle. If \(C + v\) contains another 3-cycle then these 3-cycles beat \(C, D\) by (O2). Thus, type\((v)\) is defined for all \(v \in N(C) \cap H\), and \(\|C, H\| \leq 12\). If only five vertices of \(H\) have neighbors in \(C\) then there is \(i \in [2]\) such that at most two vertices in \(H\) have type \(i\). Then \(\|\{w_i, w_{i+2}\}, H\| \leq 4\). Else every vertex in \(H\) has a neighbor in \(C\). By (2.9), \(H\) has at least four heavy vertices.

Let \(H'\) be the spanning subgraph of \(H\) with \(xy \in E(H')\) iff \(xy \in E(H)\) and \(H - \{x, y\}\) contains a 3-cycle. If \(xy \in E(H')\) then \(N(x) \cap N(y) \cap C = \emptyset\) by (O2). Now, if \(x\) and \(y\) have the same type, then they are both light. By inspection, \(H' \supseteq z_1a_1a_2z_2 + z_2z_3\).

Let type\((a_2) = i\). If \(a_2\) is heavy then its neighbors \(a', z_2, z_3\) have type \(3 - i\). Either \(z_1, a_1\) are both light or they have different types. Anyway, \(\|\{w_i, w_{i+2}\}, H\| \leq 4\). Else \(a_2\) is light. Then because there are at least four heavy vertices in \(H\), at least one of \(z_1, a_1\) is heavy and so they have different types. Also any type-\(i\) vertex in \(a', z_2, z_3\) is light, but at most one vertex of \(a, z_2, z_3\) is light because there are at most two light vertices in \(H\). Then \(\|\{w_i, w_{i+2}\}, H\| \leq 4\). By (2), there is \(i\) with \(\|w_i, w_{i+2}\), \(H\| \leq 4\); thus

\[\|\{w_i, w_{i+2}\}, V - H\| \geq (4k - 3) - 4 = 4(k - 2) + 1.\]

Now \(\|w_i, w_{i+2}\), \(D'\| \geq 5\) for some \(D' \in C - C - D\). By (2), Claim 2.20, and Lemma 2.19

\[12 \geq \|C, H\| = \|C, V - D' - (V - H - D')\| \geq 2(4k - 3) - 6 - 8(k - 3) = 12.\]

By (3), \(\|C, H\| = 12\), so each vertex in \(H\) is heavy. Thus type\((v)\) is the unique proper 2-coloring of \(H'\), and (c) follows. \(\square\)
Lemma 2.22. There exists $C^* \in \mathcal{C}$ such that $3 \leq \|\{a_1, a_2\}, C^*\| \leq 4$ and $\|\{a_1, a_2\}, D\| = 4$ for all $D \in \mathcal{C} - C^*$. If $\|\{a_1, a_2\}, C^*\| = 3$ then one of $a_1, a_2$ is low.

Proof. Suppose $\|\{a_1, a_2\}, D\| \geq 5$ for some $D \in \mathcal{C}$; set $H := H(D)$. Using Claim 2.21 choose notation so that $\|\{w_1, w_3\}, H\| \leq 4$. Now

$$\|\{w_1, w_3\}, V - H\| \geq 4k - 3 - 4 = 4(k - 2) + 1.$$  

Thus there is a cycle $B \in \mathcal{C} - D$ with $\|\{w_1, w_3\}, B\| \geq 5$; say $\|\{w_1, B\}\| = 3$. By Claim 2.20 $\|\{w_1, B\}\| \leq 6$. Note by Claim 2.21 if $\|B\| = 4$ then for an edge $z_1z_2 \in N(w_1), w_1z_1z_2w_1$ and $w_2w_3a_2w_2$ beat $B, C$ by (O2). Then $\|B\| = 3$. Using Claim 2.21(b) and Lemma 2.19

$$2(4k - 3) \leq \|C, V\| = \|C, H \cup B \cup (V - H - B)\| \leq 12 + 6 + 8(k - 3) = 2(4k - 3).$$

Thus, $\|C, D'\| \geq 4$ for all $D' \in \mathcal{C} - D$. By Lemma 2.19

$$2(4k - 3) \leq \|C, D'\| = \|\{w_1, w_3\}, D'\| = \|\{w_1, w_3\}, H\| \leq 4.$$  

By Claim 2.21(c) and Claim 2.20

$$4k - 3 \leq \|\{w_2, w_4\}, H \cup B \cup (V - H - B)\| \leq 8 + 1 + 4(k - 3) = 4k - 3,$$

and so $\|\{w_2, w_4\}, B\| = 1$. Say $\|w_2, B\| = 1$. Since $\|B\| = 3$, by Claim 2.12 $G[B \cup C - w_1]$ has two disjoint cycles that are preferible to $C, B$ by (O2). This contradiction implies $\|\{a_1, a_2\}, D\| \leq 4$ for all $D \in \mathcal{C}$. Since $\|\{a_1, a_2\}, V\| \geq 4k - 3$ and $\|\{a_1, a_2\}, R\| = 2$, we get $\|\{a_1, a_2\}, D\| \geq 3$, and equality holds for at most one $D \in \mathcal{C}$, and only if one of $a_1$ and $a_2$ is low. \hfill \Box

2.4. Completion of the proof of Theorem 1.7. For an optimal $\mathcal{C}$, let $C_i := \{D \in \mathcal{C} : |D| = i\}$ and $t_i := |C_i|$. For $C \in C_4$, let $Q_C := Q_{C}(C) := G[R(C) \cup C]$. A 3-path $R'$ is $D$-useful if $R' = R(C')$ for an optimal set $C'$ with $D \subseteq C'$; we write $D$-useful for $\{D\}$-useful.

Lemma 2.23. Let $C$ be an optimal set and $C \in C_4$. Then $Q = Q_C \in \{K_{3,4}, K_{3,4} - e\}$.

Proof. Since $C$ is optimal, $Q$ does not contain a 3-cycle. Then for all $v \in V(C)$, $N(v) \cap R$ is independent and $\|a_1, C\|, \|a_2, C\| \leq 2$. By Lemma 2.22 $\|\{a_1, a_2\}, C\| \geq 3$. Say $a_1w_1, a_2w_3 \in E$ and $\|a_2, C\| \geq 1$. Then type($a_1$) and type($a_2$) are defined.

Claim 2.24. type($a_1$) = type($a_2$).

Proof. Suppose not. Then $\|w_i, R\| \leq 1$ for all $i \in [4]$. Say $a_2w_2 \in E$. If $a_2w_2 \in E$ and $\|a_{2-j}, C\| = 2$, let $R_i = w_ia_{3-i}a_3$ and $C_i = a_{3-i}(C - w_i)a_{3-i}$ (see Figure 2.5). Then $R_i$ is $(C - C + C_i)$-useful. Let $\lambda(X)$ be the number of low vertices in $X \subseteq V$. As $Q$ does not contain a 3-cycle, $\lambda(R) + \lambda(C) \leq 2$. We claim:

$$\forall D \in \mathcal{C} - C, \|a', D\| \leq 2.$$  

Fix $D \in \mathcal{C} - C$, and suppose $\|a', D\| \geq 3$. By Claim 2.1 $\|D\| = 3$. Since

$$\|C, D\| = \|C, C'\| - \|C, C - D\|$$

$$\geq 4(2k - 1) - \lambda(C) - \|C, R\| - 8(k - 2)$$

$$= 12 - \|C, R\| - \lambda(C) \geq 6 + \lambda(R),$$

we get that $\|w_i, D\| \geq 2$ for some $i \in [4]$. If $R_i$ is defined, $R_i$ is $(C_i, D)$-useful. By Lemma 2.22 $\|\{w_i, a_i\}, D\| \leq 4$. As $\|w_i, D\| \geq 2$, $\|a', D\| \leq 2$, proving (2.10). Then $R_i$ is not defined, so $a_2$ is low with $N(a_2) \cap C = \{w_2\}$ and $\|w_2, D\| \leq 1$. Then by (2.11), $\|C - w_2, D\| \geq 6$. Note $G[a' + D] = K_4$, so for any $z \in D, D - z + a'$ is a triangle, so by (O2) the neighbors of $z$ in $C$ are independent. Then $\|C - w_2, D\| = 6$ with $N(z) \cap C = \{w_1, w_3\}$.
for every $z \in D$. Then $\|w_2, D\| = 1$, say $zw_2 \in E(G)$, and now $w_2w_3zw_2$, $w_1(D - z)w_1$ beat $C, D$ by (O2).

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{figure25.png}
\caption{Claim 2.24}
\end{figure}

If $\|a', C\| \geq 1$ then $a'w_4 \in E$ and $N(a_2) \cap C = \{w_2\}$. Now $R_2$ is $C_2$-useful, type($a'$) $\neq$ type($w_2$) with respect to $C_2$, and the middle vertex $a_2$ of $R_2$ has no neighbors in $C_2$. Thus we may assume $\|a', C\| = 0$. Then $a'$ is low:

\begin{equation}
(2.12) \quad d(a') = \|a', C \cup R\| + \|a', C - C\| \leq 0 + 2 + 2(k - 2) = 2k - 2.
\end{equation}

Thus all vertices of $C$ are high. Using Lemma 2.19 this yields:

\begin{equation}
(2.13) \quad 4 \geq \|C, R\| = \|C, V - (V - R)\| \geq 4(2k - 1) - 8(k - 1) = 4.
\end{equation}

As this calculation is tight, $d(w) = 2k - 1$ for every $w \in C$. Thus $d(a') \geq 2k - 2$, so (2.12) is tight. Hence $\|a', D\| = 2$ for all $D \in C - C$.

Pick $D = z_1 \ldots z_t z_1 \in C - C$ with $\|\{a_1, a_2\}, D\|$ maximum. By Lemma 2.22, $3 \leq \|\{a_1, a_2\}, D\| \leq 4$. Say $\|a_1, D\| \geq 2$. By (2.13), $\|C, D\| = 8$. By Lemma 2.19

\[W := G[C \cup D] \in \{K_{4,4}, K_3 \lor (K_3 + K_1), K_1 \lor K_{3,3}\}.\]

CASE 1: $W = K_{4,4}$. Then $\|D, R\| \geq 5 > |D| = 4$, so $\|z, R\| \geq 2$ for some $z \in V(D)$. Let $w \in N(z) \cap C$. Either $w$ and $z$ have a common neighbor in $\{a_1, a_2\}$ or $z$ has two consecutive neighbors in $R$. Regardless, $G[R + w + z]$ contains a 3-cycle $D'$ and $G[W - w - z]$ contains a 4-cycle $C'$. Thus $C', D'$ beats $C, D$ by (O2).

CASE 2: $W = K_3 \lor (K_3 + K_1)$. As $\|\{a', a_1\}, D\| \geq 4 > |D|$, there is $z \in V(D)$ with $D' := za'a_1z \subseteq G$. Also $W - z$ contains a 3-cycle $C'$, so $C', D'$ beats $C, D$ by (O2).

CASE 3: $W = K_1 \lor K_{3,3}$. Some $v \in V(D)$ satisfies $\|v, W\| = 6$. There is no $w \in W - v$ such that $w$ has two adjacent neighbors in $R$: else $a$ and $v$ would be contained in disjoint 3-cycles, contradicting the choice of $C, D$. Then $\|w, R\| \leq 1$ for all $w \in W - v$, because type($a_1$) $\neq$ type($a_2$). Similarly, no $z \in D - v$ has two adjacent neighbors in $R$. Thus

\[2 + 3 \leq \|a', D\| + \|a_1, a_2\|, D\| = \|R, D\| = \|R, D - v\| + \|R, v\| \leq 2 + 3,
\]

so $\|\{a_1, a_2\}, D\| = 3, R \subseteq N(v)$, and $N(a_1) \cap K_{3,3}$ is independent. By Lemma 2.22 and the maximality of $\|\{a_1, a_2\}, D\| = 3, k = 3$. Thus $G = Y_2$, a contradiction. \hfill \Box

Returning to the proof of Lemma 2.23 we have type($a_1$) = type($a_2$). Using Lemma 2.22, choose notation so that $a_1w_1, a_1w_3, a_2w_1 \in E$. Then $Q$ has bipartition $\{X, Y\}$ with $X := \{a', w_1, w_3\}$ and $Y := \{a_1, a_2, w_2, w_4\}$. The only possible nonedges between $X$ and $Y$ are $a'w_2, a'w_4$ and $a_2w_3$. Let $C' := w_1Rw_1$. Then $R' := w_2w_3w_4$ is $C'$-useful. By Lemma 2.22...
Using Lemma 2.23, one of two cases holds:

(C1) For some optimal set $C$ and $C' \in C_4$, $Q_{C'} = K_{3,4} - x_0 y_0$;

(C2) for all optimal sets $C$ and $C \in C_4$, $G[R \cup C] = K_{3,4}$.

Fix an optimal set $C$ and $C' \in C_4$, where $R = y_0 x' y$ with $d(y_0) \leq d(y)$, such that in 2.25, $Q_{C'} = K_{3,4} - x_0 y_0$. By Lemmas 2.22 and 2.23 for all $C \in C_4$, $1 \leq \|y_0, C\| \leq \|y, C\| \leq 2$ and $\|y_0, C\| = 1$ only in Case 2.25 when $C = C'$. Put $H := R \cup C_4$, $S = S(C) := N(y) \cap H$, and $T = T(C) := V(H) \setminus S$. As $\|y, R\| = 1$ and $\|y, C\| = 2$ for each $C \in C_4$, $|S| = 1 + 2t_4 = |T| - 1$.

Claim 2.25. $H$ is a bipartite graph with parts $S$ and $T$. In case 2.25 1. $H = K_{2t_4 + 1,2t_4 + 2} - x_0 y_0$; else $H = K_{2t_4 + 1,2t_4 + 2}$.

Proof. By Lemma 2.22, $\|y', S\| = \|y, T\| = \|y_0, T\| = 0$.

By Lemmas 2.22 and 2.23, $\|y_0, S\| = |S| - 1$ in 2.25 and $\|y_0, S\| = |S|$ otherwise. We claim that for every $t \in T - y_0$, $|t, S| = |S|$. This clearly holds for $y$, so take $t \in H - \{y, y_0\}$. Then $t \in C$ for some $C \in C_4$. Let $R^* := ty' x y_0$ and $C^* := y(\text{c}) t y_0$. (Note $R^*$ is a path and $C^*$ is a cycle by Lemma 2.23 and the choice of $y_0$.) Since $R^*$ is $C^*$-useful, by Lemmas 2.22 and 2.23 and by choice of $y_0$, $|t, S| = \|y, S\| = |S|$. Then in 2.25 1. $H \supseteq K_{2t_4 + 1,2t_4 + 2} - x_0 y_0$ and $x_0 y_0 \notin E(H)$; else $H \supseteq K_{2t_4 + 1,2t_4 + 2}$.

Now we easily see that if any edge exists inside $S$ or $T$, then $C_3 + (t_4 - 1)C_4 \subseteq H$, and these cycles beat $C_4$ by 2.26 2.

By Claim 2.25 all pairs of vertices of $T$ are the ends of a $C_3$-useful path. Now we use Lemma 2.22 to show that they have essentially the same degree to each cycle in $C_3$.

Claim 2.26. If $v \in T$ and $D \in C_3$ then $1 \leq \|v, D\| \leq 2$; if $\|v, D\| = 1$ then $v$ is low and for all $C \in C_3 - D$, $\|v, C\| = 2$.

Proof. By Claim 2.25 $H + x_0 y_0$ is a complete bipartite graph. Let $y_1, y_2 \in T - v$ and $u \in S - y_0$. Then $R' := y_1 u v$, $R'' := y_2 u v$, and $R''' := y_1 u y_2$ are $C_3$-useful. By Lemma 2.22

$$3 \leq \|\{v, y_1\}, D\|, \|\{v, y_2\}, D\|, \|\{v, y_1, y_2\}, D\| \leq 4.$$  

Say $\|y_1, D\| \leq 2 \leq \|y_2, D\|$. Thus

$$1 \leq \|\{v, y_1\}, D\| - \|y_1, D\| = \|v, D\| = \|\{v, y_2\}, D\| - \|y_2, D\| \leq 2.$$  

Suppose $\|v, D\| = 1$. By Claim 2.25 and Lemma 2.22 for any $v' \in T - v$,

$$4k - 3 \leq \|\{v, v'\}, H \cup (C_3 - D) \cup D\| \leq 2(2t_4 + 1) + 4(t_3 - 1) + 3 = 4k - 3.$$  

Thus for all $C \in C_3 - D_0$, $\|\{v, v'\}, C\| = 4$, and so $\|v, C\| = 2$. Hence $v$ is low. 

Next we show that all vertices in $T$ have essentially the same neighborhood in each $C \in C_3$.

Claim 2.27. Let $z \in D \in C_3$ and $v, w \in T$ with $w$ high.

1. If $zv \in E$ and $zw \notin E$ then $T - w \subseteq N(z)$.
2. $N(v) \cap D \subseteq N(w) \cap D$. 

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Proof. (1) Since \( w \) is high, Claim 2.26 implies \( \|w, D\| = 2 \). Since \( zw \notin E \), we see \( D' := w(D - z)w \) is a 3-cycle. Let \( u \in S - x_0 \). Then \( zvu = R(C') \) for some optimal set \( C' \) with \( C_3 - D + D' \subseteq C' \). By Claim 2.25, \( T(C') = S + z \) and \( S(C') = T - w \). If \((C2)\) holds, then \( T - w = S(C') \subseteq N(z) \), as desired. Suppose \((C1)\) holds, so there are \( x_0 \in S \) and \( y_0 \in T \) with \( x_0y_0 \notin E \). By Claims 2.25 and 2.26, \( d(y_0) \leq (|S| - 1) + 2t_3 = 2k - 2 \), so \( y_0 \) is low. Since \( w \) is high, we see \( y_0 \in T - w \). But now apply Claims 2.25 and 2.26 to \( T(C') \): \( d(x_0) \leq |S(C')| - 1 + 2t_3 = 2k - 2 \), and \( x_0 \) is low. As \( x_0y_0 \notin E \), this is a contradiction. Now \( T - w = S(C') \subseteq N(z) \).

(2) Suppose there exists \( z \in N(v) \cap D \setminus N(w) \). By (1), \( T - w \subseteq N(z) \). Let \( w' \in T - w \) be high. By Claim 2.26, \( \|w', D\| = 2 \). Now there exists \( z' \in N(w) \cap D \setminus N(w') \) and \( z \neq z' \). By (1), \( T - w' \subseteq N(z') \). As \( |T| \geq 4 \) and at least three of its vertices are high, there exists a high \( w'' \in T - w - w' \). Since \( w''z, w''z' \in E \), there exists \( z'' \in N(w) \cap D \setminus N(w'') \) with \( \{z, z', z''\} = V(D) \). By (1), \( T - w'' \subseteq N(z'') \). Since \( |T| \geq 4 \), there exists \( x \in T \setminus \{w, w', w''\} \). Now \( \|x, D\| = 3 \), contradicting Claim 2.26.

Let \( y_1, y_2 \in T - y_0 \) and let \( x \in S \) with \( x = x_0 \) if \( x_0y_0 \notin E \). By Claim 2.25, \( y_1, y_2, x \) is a path, and \( G - \{y_1, y_2, x\} \) contains an optimal set \( C' \). Recall \( y_0 \) was chosen in \( T \) with minimum degree, so \( y_1 \) and \( y_2 \) are high and by Claim 2.26, \( \|y_i, D\| = 2 \) for each \( i \in [2] \) and each \( D \in C_3 \). Let \( N = N(y_1) \cap \bigcup C_3 \) and \( M = \bigcup C_3 \setminus N \) (see Figure 2.6). By Claim 2.25, \( T \) is independent. By Claim 2.27, for every \( y \in T \), \( N(y) \cap \bigcup C_3 \subseteq N \), so \( E(M, T) = \emptyset \). Since \( y_2 \neq y_0 \), also \( N(y_2) \cap \bigcup C_3 = N \).

**Figure 2.6**

Claim 2.28. \( M \) is independent.

Proof. First, we show

\[
(2.14) \quad \|z, S\| \geq t_4 \quad \text{for all } z \in M.
\]

If not then there exists \( z \in D \in C_3 \) with \( \|z, S\| \leq t_4 \). Since \( \|M, T\| = \|T, T\| = 0 \),

\[
\|\{y_1, z\}, C_3\| \geq 4k - 3 - \|\{z, y_1\}, S\| \geq 4(t_4 + t_3 + 1) - 3 - (2t_4 + 1 + t_4) = t_4 + 4t_3 > 4t_3.
\]

Then there is \( D' = z'z_1z_2z' \in C_3 \) with \( \|\{z, y_1\}, D'\| \geq 5 \) and \( z' \in M \). As \( \|y_1, D\| = 2 \), \( \|z, D'\| = 3 \). Since \( D^* := zz'z_2z \) is a cycle, \( xy_2z_1 \) is \( D^* \)-useful. As \( \|z', D^*\| = 3 \), this contradicts Claim 2.26, proving (2.14).

Suppose \( zz' \in E(M) \); say \( z \in D \in C_3 \) and \( z' \in D' \in C_3 \). By (2.14), there is \( u \in N(z) \cap N(z') \cap S \). Then \( zz'u \), \( y_1(D - z)y_1 \) and \( y_2(D' - z')y_2 \) are disjoint cycles, contrary to (O1).
By Claims 2.25 and 2.28, $M$ and $T$ are independent; as remarked above $E(M,T) = \emptyset$. Then $M \cup T$ is independent. This contradicts (H3), since
\[|G| - 2k + 1 = 3t_3 + 4t_4 + 3 - 2(t_3 + t_4 + 1) + 1 = t_3 + 2t_4 + 2 = |M \cup T| \leq \alpha(G).\]
The proof of Theorem 1.7 is now complete.  

3. The case $k = 2$

Lovász [22] observed that any (simple or multi-) graph can be transformed into a multigraph with minimum degree at least 3, without affecting the maximum number of disjoint cycles in the graph, by using a sequence of operations of the following three types: (i) deleting a bud; (ii) suppressing a vertex $v$ of degree 2 that has two neighbors $x$ and $y$, i.e., deleting $v$ and adding a new (possibly parallel) edge between $x$ and $y$; and (iii) increasing the multiplicity of a loop or edge with multiplicity 2. Here loops and two parallel edges are considered cycles, so forests have neither. Also $K_s$ and $K_{s,t}$ denote simple graphs. Let $W^*_s$ denote a wheel on $s$ vertices whose spokes, but not outer cycle edges, may be multiple. The following theorem characterizes those multigraphs that do not have two disjoint cycles.

Theorem 3.1 (Lovász [22]). Let $G$ be a multigraph with $\delta(G) \geq 3$ and no two disjoint cycles. Then $G$ is one of the following: (1) $K_5$, (2) $W^*_s$, (3) $K_{3,|G|−3}$ together with a multigraph on the vertices of the (first) 3-class, and (4) a forest $F$ and a vertex $x$ with possibly some loops at $x$ and some edges linking $x$ to $F$.

Let $G$ be the class of simple graphs $G$ with $|G| \geq 6$ and $\sigma_2(G) \geq 5$ that do not have two disjoint cycles. Fix $G \in G$. A vertex in $G$ is low if its degree is at most 2. The low vertices form a clique $Q$ of size at most 2—if $|Q| = 3$, then $Q$ is a component-cycle, and $G - Q$ has another cycle. By Lovász’s observation, $G$ can be reduced to a graph $H$ of type (1–4). Reversing this reduction, $G$ can be obtained from $H$ by adding buds and subdividing edges. Let $Q' := V(G) \setminus V(H)$. It follows that $Q \subseteq Q'$. If $Q' \neq Q$, then $Q$ consists of a single leaf in $G$ with a neighbor of degree 3, so $G$ is obtained from $H$ by subdividing an edge and adding a leaf to the degree-2 vertex. If $Q' = Q$, then $Q$ is a component of $G$, or $G = H + Q + e$ for some edge $e \in E(H,Q)$, or at least one vertex of $Q$ subdivides an edge $e \in E(H)$. In the last case, when $|Q| = 2$, $e$ is subdivided twice by $Q$. As $G$ is simple, $H$ has at most one multiple edge, and its multiplicity is at most 2.

In case (4), because $\delta(H) \geq 3$, either $F$ has at least two buds, each linked to $x$ by multiple edges, or $F$ has one bud linked to $x$ by an edge of multiplicity at least 3. This case cannot arise from $G$. Also, $\delta(H) = 3$, unless $H = K_5$, in which case $\delta(H) = 4$. Then $Q$ is not an isolated vertex, lest deleting $Q$ leave $H$ with $\delta(H) \geq 5 > 4$; and if $Q$ has a vertex of degree 1 then $H = K_5$. Else all vertices of $Q$ have degree 2, and $Q$ consists of the subdivision vertices of one edge of $H$. We have the following lemma.

Lemma 3.2. Let $G$ be a graph with $|G| \geq 6$ and $\sigma_2(G) \geq 5$ that does not have two disjoint cycles. Then $G$ is one of the following (see Figure 3.1):

(a) $K_5 + K_2$;
(b) $K_5$ with a pendant edge, possibly subdivided;
(c) $K_5$ with one edge subdivided and then a leaf added adjacent to the degree-2 vertex;
(d) a graph $H$ of type (1–3) with no multiple edge, and possibly one edge subdivided once or twice, and if $|H| = 6 - i$ with $i \geq 1$ then some edge is subdivided at least $i$ times;
(e) a graph $H$ of type (2) or (3) with one edge of multiplicity two, and one of its parallel parts is subdivided once or twice—twice if $|H| = 4$.

Figure 3.1. Theorem 3.2

4. Connections to Equitable Coloring

A proper vertex coloring of a graph $G$ is equitable if any two color classes differ in size by at most one. In 1970 Hajnal and Szemerédi proved:

**Theorem 4.1** ([10]). *Every graph $G$ with $\Delta(G) + 1 \leq k$ has an equitable $k$-coloring.*

For a shorter proof of Theorem 4.1 see [18]; for an $O(k|G|^2)$-time algorithm see [17].

Motivated by Brooks’ Theorem, it is natural to ask which graphs $G$ with $\Delta(G) = k$ have equitable $k$-colorings. Certainly such graphs are $k$-colorable. Also, if $k$ is odd then $K_{k,k}$ has no equitable $k$-coloring. Chen, Lih, and Wu [3] conjectured (in a different form) that these are the only obstructions to an equitable version of Brooks’ Theorem:

**Conjecture 4.2** ([3]). *If $G$ is a graph with $\chi(G), \Delta(G) \leq k$ and no equitable $k$-coloring then $k$ is odd and $K_{k,k} \subseteq G$.*

In [3], Chen, Lih, and Wu proved Conjecture 4.2 holds for $k = 3$. By a simple trick, it suffices to prove the conjecture for graphs $G$ with $|G| = ks$. Combining the results of the two papers [14] and [15], we have:

**Theorem 4.3.** *Suppose $G$ is a graph with $|G| = ks$. If $\chi(G), \Delta(G) \leq k$ and $G$ has no equitable $k$-coloring, then $k$ is odd and $K_{k,k} \subseteq G$ or both $k \geq 5$ [14] and $s \geq 5$ [15].*
A graph $G$ is $k$-equitable if $|G| = ks$, $\chi(G) \leq k$ and every proper $k$-coloring of $G$ has $s$ vertices in each color class. The following strengthening of Conjecture 4.2, if true, provides a characterization of graphs $G$ with $\chi(G), \Delta(G) \leq k$ that have an equitable $k$-coloring.

**Conjecture 4.4** ([13]). Every graph $G$ with $\chi(G), \Delta(G) \leq k$ has no equitable $k$-coloring if and only if $k$ is odd and $G = H + K_{k,k}$ for some $k$-equitable graph $H$.

The next theorem collects results from [13]. Together with Theorem 4.3 it yields Corollary 4.6.

**Theorem 4.5** ([13]). Conjecture 4.2 is equivalent to Conjecture 4.4. Indeed, for any $k_0$ and $n_0$, Conjecture 4.2 holds for $k \leq k_0$ and $|G| \leq n_0$ if and only if Conjecture 4.4 holds for $k \leq k_0$ and $|G| \leq n_0$.

**Corollary 4.6.** A graph $G$ with $|G| = 3k$ and $\chi(G), \Delta(G) \leq k$ has no equitable $k$-coloring if and only if $k$ is odd and $G = K_{k,k} + K_k$.

We are now ready to complete our answer to Dirac’s question for simple graphs.

**Proof of Theorem 4.6.** Assume $k \geq 2$ and $\delta(G) \geq 2k - 1$. It is apparent that if any of (i), (H3), or (H4) in Theorem 4.3 fail, then $G$ does not have $k$ disjoint cycles. Now suppose $G$ satisfies (i), (H3), and (H4). If $k = 2$ then $|G| \geq 6$ and $\delta(G) \geq 3$. Thus $G$ has no subdivided edge, and only (d) of Lemma 3.2 is possible. By (i), $G \neq K_5$; by (H4), $G$ is not a wheel; and by (H3), $G$ is not type (3) of Theorem 3.1. Then $G$ has 2 disjoint cycles. Finally, suppose $k \geq 3$. Since $G$ satisfies (ii), we see $G \notin \{Y_1, Y_2\}$ and $G$ satisfies (H2). If $|G| \geq 3k + 1$, then $G$ has $k$ disjoint cycles by Theorem 4.7. Otherwise, $|G| = 3k$ and $G$ has $k$ disjoint cycles if and only if its vertices can be partitioned into disjoint $K_3$’s. This is equivalent to $\overline{G}$ having an equitable $k$-coloring. By (ii), $\Delta(\overline{G}) \leq k$, and by (H3), $\omega(\overline{G}) \leq k$. Then by Brooks’ Theorem, $\chi(\overline{G}) \leq k$. By (H4) and Corollary 4.6 $\overline{G}$ has an equitable $k$-coloring. □

Next we turn to Ore-type results on equitable coloring. To complement Theorem 4.7 we need a theorem that characterizes when a graph $G$ with $|G| = 3k$ that satisfies (H2) and (H3) has $k$ disjoint cycles, or equivalently, when its complement $\overline{G}$ has an equitable coloring. The complementary version of $\sigma_2(G)$ is the maximum Ore-degree $\theta(H) := \max_{xy \in E(H)}(d(x) + d(y))$. Then $\theta(\overline{G}) = 2|G| - \sigma_2(G) - 2$, and if $|G| = 3k$ and $\sigma_2(G) \geq 4k - 3$ then $\theta(\overline{G}) \leq 2k + 1$. Also, if $G$ satisfies (H3) then $\omega(\overline{G}) \leq k$. This would correspond to an Ore-Brooks-type theorem on equitable coloring.

Several papers, including [11] [12] [21], address equitable colorings of graphs $G$ with $\theta(G)$ bounded from above. For instance, the following is a natural Ore-type version of Theorem 4.1.

**Theorem 4.7** ([11]). Every graph $G$ with $\theta(G) \leq 2k - 1$ has an equitable $k$-coloring.

Even for proper (not necessarily equitable) coloring, an Ore-Brooks-type theorem requires forbidding some extra subgraphs when $\theta$ is 3 or 4. It was observed in [12] that for $k = 3, 4$ there are graphs for which $\theta(G) \leq 2k + 1$ and $\omega(G) \leq k$, but $\chi(G) \geq k + 1$. The following theorem was proved for $k \geq 6$ in [12] and then for $k \geq 5$ in [21].

**Theorem 4.8.** Let $k \geq 5$. If $\omega(G) \leq k$ and $\theta(G) \leq 2k + 1$, then $\chi(G) \leq k$.

In the subsequent paper [16] we prove an analog of Theorem 4.7 for $3k$-vertex graphs.

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References


