Recent results on disjoint and longest cycles in graphs

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Corrádi–Hajnal Theorem

It was a conjecture by Erdős:

**Theorem 1** [Corrádi and Hajnal, 1963, first version 1961]: Let $k \geq 1$, $n \geq 3k$ and let $H$ be an $n$-vertex graph with $\delta(H) \geq 2k$. Then $H$ contains $k$ vertex-disjoint cycles.

**Remark:** In fact, they showed that one can have the length of each of the $k$ cycles be at most $\lceil n/k \rceil$.

**Corollary 2** [Corrádi and Hajnal]: Let $n = 3k$ and $H$ be an $n$-vertex graph with $\delta(H) \geq 2k$. Then $H$ contains $k$ vertex-disjoint triangles.

Both restrictions are sharp.
Examples

$k=3$

Figure: Graphs with mindegree 5 with no 3 disjoint cycles.
Refinements

\[ \Theta(G) = \min_{xy \notin E(G)} d(x) + d(y). \]

**Theorem 3** [Enomoto 1998, Wang 1999]: Let \( k \geq 1, n \geq 3k \) and let \( H \) be an \( n \)-vertex graph with \( \Theta(H) \geq 4k - 1 \). Then \( H \) contains \( k \) vertex-disjoint cycles.

**Theorem 4** [Aigner and Brandt 1993, Alon and Fisher 1996]: Let \( n \geq 3 \) and \( H \) be an \( n \)-vertex graph with \( \delta(H) \geq 2n/3 \). Then \( H \) contains each 2-factor.

**Theorem 5** [Fan and Kierstead 1996]: Let \( n \geq 3 \) and \( H \) be an \( n \)-vertex graph with \( \delta(H) \geq 2n/3 - 1 \). Then \( H \) contains the square of the \( n \)-vertex path.
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Theorem 3 [Enomoto 1998, Wang 1999]: Let \( k \geq 1, n \geq 3k \) and let \( H \) be an \( n \)-vertex graph with \( \Theta(H) \geq 4k - 1 \). Then \( H \) contains \( k \) vertex-disjoint cycles.

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Theorem 5 [Fan and Kierstead 1996]: Let \( n \geq 3 \) and \( H \) be an \( n \)-vertex graph with \( \delta(H) \geq \frac{2n-1}{3} \). Then \( H \) contains the square of the \( n \)-vertex path.

Theorem 6 [A.K. and Yu]: Let \( n \geq 3 \) and \( H \) be an \( n \)-vertex graph with \( \Theta(H) \geq \frac{4n}{3} - 1 \). Then \( H \) contains each 2-factor.
Hajnal-Szemerédi Theorem

This also was a conjecture by Erdős:

**Theorem 7 [Hajnal and Szemerédi 1970]:** If $n = sk$ and for an $n$-vertex $G$, $\delta(G) \geq (s - 1)k$, then $G$ contains $k$ disjoint $K_s$.

**Theorem 8 [Kierstead and A.K. 2008]:** If $n = sk$ and for an $n$-vertex $G$, $\Theta(G) \geq 2(s - 1)k - 1$, then $G$ contains $k$ disjoint $K_s$. 
Dirac’s question

In 1963 Dirac described all 3-connected multigraphs that do not have two disjoint cycles.

Also in 1963 Lovász described all multigraphs that do not have two disjoint cycles

Dirac asked:
Question 1 [Dirac 1963]: Which \((2k - 1)\)-connected multigraphs do not have \(k\) disjoint cycles?
Dirac-Erdős Theorems

Theorem 9 [Dirac and Erdős, 1963]: For $k \geq 3$, every graph in which the number $|V_{\geq 2k}|$ of the vertices with degree at least $2k$ exceeds the number $|V_{\leq 2k-2}|$ of the vertices with degree at most $2k - 2$ by at least $k^2 + 2k - 4$ contains $k$ disjoint cycles.

Theorem 10 [Dirac and Erdős, 1963]: For $k \geq 3$, every planar graph in which the number $|V_{\geq 2k}|$ of the vertices with degree at least $2k$ exceeds the number $|V_{\leq 2k-2}|$ of the vertices with degree at most $2k - 2$ by at least $5k - 7$ contains $k$ disjoint cycles.

These theorems were inspired by the Corrádi-Hajnal Theorem, but appeared in print earlier.
Figure: A graph with no $k$ disjoint cycles and $|V_{\geq 2k}| - |V_{\leq 2k-2}| = 2k - 1$. 
Theorem 11 [Kierstead, A.K., and McConvey]: Let $k \geq 2$ be an integer and $G$ be a graph such that $|G| \geq 3k$. Let $t$ be the maximum number of disjoint triangles contained in $G$. If

$$|V_{\geq 2k}| - |V_{\leq 2k-2}| \geq 2k + t,$$

then $G$ contains $k$ disjoint cycles.

Figure: A graph with no $k$ disjoint cycles and $|V_{\geq 2k}| - |V_{\leq 2k-2}| = 3k - 2.$
Theorem 12 [Kierstead, A.K., and McConvey]: Let $k \geq 2$ be an integer and $G$ be a planar graph such that $|G| \geq 3k$. If

$$|V_{\geq 2k}| - |V_{\leq 2k-2}| \geq 2k,$$

then $G$ contains $k$ disjoint cycles.

Figure: A $4k$-vertex graph with no $k$ disjoint cycles and $|V_{\geq 2k}| - |V_{\leq 2k-2}| = 2k$. 
Theorem 13 [Kierstead, A.K., and McConvey]: Let $k \geq 2$ be an integer and $G$ be a graph with $|G| \geq 19k$. If

$$|V_{\geq 2k}| - |V_{\leq 2k-2}| \geq 2k,$$

then $G$ contains $k$ disjoint cycles.

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Theorem 13 [Kierstead, A.K., and McConvey]: Let \( k \geq 2 \) be an integer and \( G \) be a graph with \( |G| \geq 19k \). If

\[
|V_{\geq 2k}| - |V_{\leq 2k-2}| \geq 2k,
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then \( G \) contains \( k \) disjoint cycles.

Figure: A \( 4k \)-vertex graph with no \( k \) disjoint cycles and \( |V_{\geq 2k}| - |V_{\leq 2k-2}| = 2k \).

**Question:** Is it true that for each \( n \geq 4k + 1 \) every graph with

\[
|V_{\geq 2k}| - |V_{\leq 2k-2}| \geq 2k
\]

has \( k \) disjoint cycles?
Erdős–Gallai Theorems

Theorem 14 [Erdős and Gallai, 1959]: Let \( n \geq k \geq 2 \) and \( G \) be an \( n \)-vertex graph with more than \( \frac{1}{2} (k - 2)n \) edges. Then \( G \) contains a \( k \)-vertex path \( P_k \).

Theorem 15 [Erdős and Gallai, 1959]: Let \( n' \geq k' \geq 3 \) and \( G \) be an \( n' \)-vertex graph with more than \( \frac{1}{2} (k' - 1)(n' - 1) \) edges. Then \( G \) contains a cycle of length at least \( k \).
Erdős–Gallai Theorems

Theorem 14 [Erdős and Gallai, 1959]: Let $n \geq k \geq 2$ and $G$ be an $n$-vertex graph with more than $\frac{1}{2}(k - 2)n$ edges. Then $G$ contains a $k$-vertex path $P_k$.

Theorem 15 [Erdős and Gallai, 1959]: Let $n' \geq k' \geq 3$ and $G$ be an $n'$-vertex graph with more than $\frac{1}{2}(k' - 1)(n' - 1)$ edges. Then $G$ contains a cycle of length at least $k$.

Theorem 14 follows from Theorem 15 for $n' = n + 1$ and $k' = k + 1$. Indeed, if an $n$-vertex graph $G$ with $e(G) > \frac{1}{2}(k - 2)n$ has no $k$-vertex path, consider $G'$ obtained from $G$ by adding a new vertex $v$ adjacent to all vertices of $G$. Then $G'$ has $n' = n + 1$ vertices, $e(G') = n + e(G) > \frac{1}{2}kn = \frac{1}{2}(k' - 1)(n' - 1)$, but $G'$ has no cycle with at least $k'$ vertices, a contradiction to Theorem 15.
Erdős–Gallai Theorems, II

Theorem 14 [Erdős and Gallai, 1959]: Let \( n \geq k \geq 2 \) and \( G \) be an \( n \)-vertex graph with more than \( \frac{1}{2}(k - 2)n \) edges. Then \( G \) contains a \( k \)-vertex path \( P_k \).

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Theorem 14 is sharp for \( n \) divisible by \( k - 1 \): Let \( G_1 \) be a disjoint union of \( K_{k-1} \)s.

Theorem 15 is sharp for \( n' - 1 \) divisible by \( k' - 2 \): Let \( G_2 \) be obtained from a disjoint union of \( K_{k'-2} \)s by adding an all-adjacent vertex.
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Theorem 14 [Erdős and Gallai, 1959]: Let \( n \geq k \geq 2 \) and \( G \) be an \( n \)-vertex graph with more than \( \frac{1}{2}(k - 2)n \) edges. Then \( G \) contains a \( k \)-vertex path \( P_k \).

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Erdős and Gallai also gave an exact upper bound on \( e(G) \) of a connected \( n \)-vertex graph \( G \) with no \( k \)-vertex path for even \( k \) and \( n \geq \frac{k^2}{4} - k + 7 \). Moreover, they also described the extremal graphs.
Theorems 14 and 15 were refined:

1. Faudree and Schelp and independently Kopylov found the exact value of $\text{ex}(n, P_k)$ for all pairs $(n, k)$. Moreover, Faudree and Schelp described all extremal examples.

2. Woodall found the exact values for Theorem 15 for all pairs $(n', k')$. Moreover, he has extended the Erdős-Gallai theorem on paths in connected graphs to all $n \geq 3k/2$ and has found the exact maximum number of edges in $n'$-vertex 2-connected graphs with no cycles of length at least $k'$ for $n' \geq 3k'/2$. Voss also announced part of these results.

3. Kopylov fixed all the four problems for all values of $n, k, n'$ and $k'$. 
A construction

Let \( n \geq k, \frac{k}{2} > a \geq 1 \). Define the \( n \)-vertex \( H_{n,k,a} \):

\[
V(H_{n,k,a}) = A \cup B \cup C, \text{ where } |A| = a, |B| = n - k + a, |C| = k - 2a.
\]

\[H_{n,k,a}[A \cup C] = K_{k-a}, \quad H_{n,k,a}[A \cup B] = K_{n-k+2a} - E(K_{n-k+a})\]

and no edges between \( B \) and \( C \).

Figure: Graph \( H_{11,11,3} \).

Let

\[h(n, k, a) = e(H_{n,k,a}) = \binom{k-a}{2} + a(n - k + a).\]
Kopylov’s Theorem

Theorem 16 [Kopylov, 1977] Let $n \geq k \geq 5$ and $t = \lfloor \frac{k-1}{2} \rfloor$. If $G$ is an $n$-vertex 2-connected graph with no cycle of length at least $k$, then

$$e(G) \leq \max\{h(n, k, 2), h(n, k, t)\}$$

(1)

with equality only if $G = H_{n,k,2}$ or $G = H_{n,k,t}$.

All three other results (exact forms of Theorems 14 and 15 and the exact bound for paths in connected graphs) follow from this theorem.
Theorem 17 [Füredi, A. K., and Verstraëte] Let $t \geq 2$ and $n \geq 3t$ and $k \in \{2t + 1, 2t + 2\}$. Let $G$ be a 2-connected $n$-vertex graph containing no cycle of length at least $k$. Then

$$e(G) \leq h(n, k, t - 1)$$

unless

(a) $k = 2t + 1$, $k \neq 7$, and $G \subseteq H_{n,k,t}$

or

(b) $k = 2t + 2$ or $k = 7$, and $G - A$ is a star forest for some $A \subseteq V(G)$ of size at most $t$. 
Theorem 17 [Füredi, A. K., and Verstraëte] Let $t \geq 2$ and $n \geq 3t$ and $k \in \{2t + 1, 2t + 2\}$. Let $G$ be a 2-connected $n$-vertex graph containing no cycle of length at least $k$. Then $e(G) \leq h(n, k, t - 1)$ unless

(a) $k = 2t + 1$, $k \neq 7$, and $G \subseteq H_{n,k,t}$ or
(b) $k = 2t + 2$ or $k = 7$, and $G - A$ is a star forest for some $A \subset V(G)$ of size at most $t$.

The condition $e(G) > h(n, k, t - 1)$ is best possible, since $H_{n,k,t-1}$ contains no cycle of length at least $k$, is not a subgraph of $H_{n,k,t}$, and $H_{n,2t+2,t-1} - A$ has a cycle for every $A \subset V(H_{n,2t+2,t-1})$ with $|A| = t$. 
Since
\[ h(n, 2t + 2, t) = \binom{t}{2} + t(n - t) + 1 = h(n, 2t + 1, t) + 1 \]
and
\[ h(n, 2t + 2, t - 1) = \binom{t}{2} + (t-1)(n-t) + 6 = h(n, 2t + 1, t - 1) + 3, \]
the difference between Kopylov’s bound and the bound in Theorem 17 is
\[ h(n, k, t) - h(n, k, t - 1) = \begin{cases} n - t - 3 & \text{if } k = 2t + 1 \\ n - t - 5 & \text{if } k = 2t + 2. \end{cases} \] (2)

So, for a fixed \( k \), the difference in (2) divided by \( h(n, k, t) \) does not tend to 0 when \( n \to \infty \).
Paths

Similarly to deducing Theorem 14 from Theorem 15, Theorem 17 yields:

Theorem 18 [Füredi, A. K., and Verstraëte]
Let $t \geq 2$ and $n \geq 3t - 1$ and $k \in \{2t, 2t + 1\}$, and let $G$ be a connected $n$-vertex graph containing no $k$-vertex path. Then $e(G) \leq h(n, k - 1, t - 2)$ unless

(a) $k = 2t$, $k \neq 6$, and $G \subseteq H_{n,k-1,t-1}$ or
(b) $k = 2t + 1$ or $k = 6$, and $G - A$ is a star forest for some $A \subset V(G)$ of size at most $t - 1$.

Proof: Let $G'$ be obtained from an $n$-vertex connected $G$ with $e(G) > h(n, k - 1, t - 2)$ by adding an all-adjacent $v$. Then $G'$ is 2-connected and $e(G') > h(n, k - 1, t - 2) + n = h(n + 1, k + 1, t - 1)$. So, we can apply Theorem 17 to $G'$. 
Similarly to deducing Theorem 14 from Theorem 15, Theorem 17 yields:

**Theorem 18 [Füredi, A. K., and Verstraëte]**

Let \( t \geq 2 \) and \( n \geq 3t - 1 \) and \( k \in \{2t, 2t + 1\} \), and let \( G \) be a connected \( n \)-vertex graph containing no \( k \)-vertex path. Then \( e(G) \leq h(n, k - 1, t - 2) \) unless

(a) \( k = 2t, k \neq 6 \), and \( G \subseteq H_{n,k-1,t-1} \) or

(b) \( k = 2t + 1 \) or \( k = 6 \), and \( G - A \) is a star forest for some \( A \subset V(G) \) of size at most \( t - 1 \).

**Proof:** Let \( G' \) be obtained from an \( n \)-vertex connected \( G \) with \( e(G) > h(n, k - 1, t - 2) \) by adding an all-adjacent \( v \). Then \( G' \) is 2-connected and

\[ e(G') > h(n, k - 1, t - 2) + n = h(n + 1, k + 1, t - 1). \]

So, we can apply Theorem 17 to \( G' \).
Some graph classes

Let $G_1(n, k) = \{H_{n,k,t}\}$. Each $G \in G_2(n, k)$ is defined by a partition $V(G) = A \cup B \cup J$, $|A| = t$ and a pair $a_1 \in A$, $b_1 \in B$ such that $G[A] = K_t$, $G(A, B)$ is a complete bipartite graph and for every $c \in J$ one has $N(c) = \{a_1, b_1\}$. Every member of $G \in G_3(n, k)$ is defined by a partition $V(G) = A \cup B \cup J$, $|A| = t$ such that $G[A] = K_t$, $G(A, B)$ is a complete bipartite graph, and

- $G[J]$ has more than one component;
- each component of $G[J]$ is a star with at least two vertices;
- there is a 2-element subset $A'$ of $A$ such that $N(J) \cap (A \cup B) = A'$;
- for each component $S$ of $G[J]$ with at least 3 vertices, all leaves of $S$ are adjacent to the same vertex $a(S) \in A'$. 
The class $G_4(n, k)$ is empty unless $k = 10$. Each member of $G_4(n, 10)$ has a 3-vertex set $A$ such that $G[A] = K_3$ and $G - A$ is a star forest such that if a component $S$ of $G - A$ has more than two vertices then all its leaves are adjacent to the same vertex $a(S) \in A$.

Classes $G_2(n, k), G_3(n, k)$ and $G_4(n, 10)$. 
Theorem 19 Let $k \geq 9$, $n \geq \frac{3k}{2}$ and $t = \left\lfloor \frac{k-1}{2} \right\rfloor$. Let $G$ be an $n$-vertex 2-connected graph with no cycle of length at least $k$. Then $e(G) \leq h(n, k, t - 1)$ or $G$ is a subgraph of a graph in $\mathcal{G}(n, k)$, where

1. if $k$ is odd, then $\mathcal{G}(n, k) = \mathcal{G}_1(n, k) = \{H_{n,k,t}\}$;
2. if $k$ is even and $k \neq 10$, then $\mathcal{G}(n, k) = \mathcal{G}_1(n, k) \cup \mathcal{G}_2(n, k) \cup \mathcal{G}_3(n, k)$;
3. if $k = 10$, then $\mathcal{G}(n, k) = \mathcal{G}_1(n, 10) \cup \mathcal{G}_2(n, 10) \cup \mathcal{G}_3(n, 10) \cup \mathcal{G}_4(n, 10)$. 
We use contractions. A simple observation: If $c(G) < k$, then for any $G'$ obtained from $G$ by contracting an edge, $c(G') < k$. 

Lemma 20: Let $n \geq 4$ and let $G$ be an $n$-vertex 2-connected graph. Let $v \in V(G)$. Then there is $w \in N(v)$ such that $G_{vw}$ is 2-connected.

Roughly speaking, for $n > 3k/2$, we start from an $n$-vertex $G$ with $e(G) > h(n, k, t-1)$ and contract edges keeping 2-connectedness and decreasing the number of edges by at most $t-1$ at each step. We stop when either each edge is in many triangles or the number of vertices becomes $k$. Then we find a special structure in the resulting graph $G_m$ and show that after uncontracting edges we still find a suitable subgraph in the original $G$. Finally, we show that each $n$-vertex $G$ without long cycles but with $e(G) > h(n, k, t-1)$ containing special subgraphs satisfies the theorem.
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Small $n$

Theorem 21 [Füredi, A. K., Luo, and Verstraëte]
Let $t \geq 4$, $k \in \{2t + 1, 2t + 2\}$ and $n \geq k$. Let $G$ be a 2-connected $n$-vertex graph containing no cycle of length at least $k$. Then $e(G) \leq \max\{h(n, k, t-1), h(n, k, 3)\}$ unless

(a) $k = 2t + 1$ and $G \subseteq H_{n,k,t}$ or $G \subseteq H_{n,k,2}$ or

(b) $k = 2t + 2$ and $G - A$ is a star forest for some $A \subset V(G)$ of size at most $t$. 

The idea is to prove first the case $n \leq 3k/2$: we try to contract edges as above, and if we come to a graph with few edges, then we return to the start and do Kopylov's disintegration with a special order of disintegrated vertices. When the case $n \leq 3k/2$ is resolved, we use a simple induction for larger $n$. 

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