THE STRUCTURE OF LARGE INTERSECTING FAMILIES

ALEXANDR KOSTOCHKA AND DHRUV MUBAYI

(Communicated by Patricia L. Hersh)

Abstract. A collection of sets is intersecting if every two members have nonempty intersection. We describe the structure of intersecting families of r-sets of an n-set whose size is quite a bit smaller than the maximum \((\binom{n-1}{r-1})\) given by the Erdős-Ko-Rado Theorem. In particular, this extends the Hilton-Milner theorem on nontrivial intersecting families and answers a recent question of Han and Kohayakawa for large n. In the case \(r = 3\) we describe the structure of all intersecting families with more than 10 edges. We also prove a stability result for the Erdős matching problem. Our short proofs are simple applications of the Delta-system method introduced and extensively used by Frankl since 1977.

1. Introduction

An \(r\)-uniform hypergraph \(H\), or simply an \(r\)-graph, is a family of \(r\)-element subsets of a finite set. We associate an \(r\)-graph \(H\) with its edge set and call its vertex set \(V(H)\). Say that \(H\) is intersecting if \(A \cap B \neq \emptyset\) for all \(A, B \in F\). A matching in \(H\) is a collection of pairwise disjoint sets from \(H\). A vertex cover (henceforth cover) of \(H\) is a set of vertices intersecting every edge of \(H\). Write \(\nu(H)\) for the size of a maximum matching and \(\tau(H)\) for the size of a minimum cover of \(H\). Say that \(H\) is trivial or a star if \(\tau(H) = 1\), otherwise call \(H\) nontrivial.

A fundamental problem in the extremal theory of finite sets is to determine the maximum size of an \(n\)-vertex \(r\)-graph \(H\) with \(\nu(H) \leq s\). The case \(s = 1\) is when \(H\) is intersecting, and in this case the Erdős-Ko-Rado Theorem \([3]\) states that the maximum is \(\binom{n-1}{r-1}\) for \(n \geq 2r\) and if \(n > 2r\), then equality holds only if \(\tau(H) = 1\). More generally, Erdős \([2]\) proved the following.

Theorem 1 (Erdős \([2]\)). For \(r \geq 2\), \(s \geq 1\) and \(n\) sufficiently large, every \(n\)-vertex \(r\)-graph \(H\) with \(\nu(H) \leq s\), satisfies

\[
|H| \leq cm(n, r, s) := \binom{n}{r} - \binom{n-s}{r} \sim s \binom{n}{r-1},
\]

and if equality in (1) holds, then \(H\) is the \(r\)-graph \(EM(n, r, s)\) described below.

Received by the editors February 3, 2016 and, in revised form, February 4, 2016 and July 20, 2016.

2010 Mathematics Subject Classification. Primary 05B07, 05C65, 05C70, 05D05, 05D15.

The research of the first author was supported in part by NSF grants DMS-1266016 and DMS-1600592 and by grants 15-01-05867 and 16-01-00499 of the Russian Foundation for Basic Research.

The research of the second author was partially supported by NSF grant DMS-1300138.
Construction 1. Let $EM(n, r, s)$ be the $n$-vertex $r$-graph that has $s$ special vertices $x_1, \ldots, x_s$ and the edge set consists of all the $r$-sets intersecting \{x_1, \ldots, x_s\}. In particular, $EM(n, r, 1)$ is a full star.

There has been a lot of recent activity on Theorem 1 for small $n$ (see, e.g., [10][11][16][17]).


\textbf{Theorem 2} (Hilton-Milner [15], Proposition 7). Suppose that $2 \leq r \leq n/2$ and $|H|$ is an $n$-vertex intersecting $r$-graph with $\tau(H) \geq 2$. Then

\begin{equation}
|H| \leq hm(n, r) := \binom{n-1}{r-1} - \binom{n-r-1}{r-1} + 1 \sim r \binom{n}{r-2}.
\end{equation}

Moreover, if $4 \leq r < n/2$ and (2) holds with equality, then $H$ is the $r$-graph $HM(n, r)$ described below.

\textbf{Construction 2.} For $n \geq 2r$, let $HM(n, r)$ be the following $r$-graph on $n$ vertices: Choose an $r$-set $X = \{x_1, \ldots, x_r\}$ and a special vertex $x \notin X$, and let $HM(n, r)$ consist of the set $X$ and all $r$-sets containing $x$ and a vertex of $X$.

Observe that $HM(n, r)$ is intersecting, $\tau(HM(n, r)) = 2$, and $|HM(n, r)| = hm(n, r)$. Bollobás, Daykin and Erdős [1] extended Theorem 2 to $r$-graphs with matching number $s$ in the way Theorem 1 extends the Erdős-Ko-Rado Theorem.

\textbf{Theorem 3} (Bollobás-Daykin-Erdős [1], Theorem 1). Suppose $r \geq 2$, $s \geq 1$ and $n > 2r^3s$. If $H$ is an $n$-vertex $r$-graph with $\nu(H) \leq s$ and $|H| > em(n, r, s-1) + hm(n-s+1, r)$, then $H \subseteq EM(n, r, s)$.

The bound of Theorem 3 is also sharp: take a copy of $HM(n-s+1, r)$, add an extra set $S$ of $s-1$ vertices and all edges intersecting with $S$. Han and Kohayakawa [14] refined Theorem 2 using the following construction.

\textbf{Construction 3.} For $r \geq 3$, the $n$-vertex $r$-graph $HM'(n, r)$ has $r+2$ distinct special vertices $x, x_1, \ldots, x_{r-1}, y_1, y_2$ and all edges $e$ such that

1) $\{x, x_i\} \subseteq e$ for any $i \in [r-1]$, or
2) $\{x, y_1, y_2\} \subseteq e$, or
3) $e = \{x_1, \ldots, x_{r-1}, y_1\}$, or $e = \{x_1, \ldots, x_{r-1}, y_2\}$.

Note that $HM'(n, r)$ is intersecting, $\tau(HM'(n, r)) = 2$, and $HM'(n, r) \not\subseteq HM(n, r)$. Let $hm'(n, r) = |HM'(n, r)|$ so that

\begin{equation}
hm'(n, r) = \binom{n-1}{r-1} - \binom{n-r}{r-1} + \binom{n-r-2}{r-3} + 2 \sim (r-1) \binom{n}{r-2}.
\end{equation}

The result of [14] for $r \geq 5$ is:

\textbf{Theorem 4} (Han-Kohayakawa [14]). Let $r \geq 5$ and $n > 2r$. If $H$ is an $n$-vertex intersecting $r$-graph, $\tau(H) \geq 2$ and $|H| \geq hm'(n, r)$, then $H \subseteq HM(n, r)$ or $H = HM'(n, r)$.

They also resolved the cases $r = 4$ and $r = 3$, where the statements are similar but somewhat more involved.

For large $n$, Frankl [8] gave an exact upper bound on the size of intersecting $n$-vertex $r$-graphs $H$ with $\tau(H) \geq 3$. He introduced the following family. We write $A + a$ to mean $A \cup \{a\}$.
Construction 4 ([8]). The vertex set $[n]$ of the $n$-vertex $r$-graph $FP(n, r)$ contains a special subset $X = \{x\} \cup Y \cup Z$ with $|X| = 2r$ such that $|Y| = r$, $|Z| = r - 1$, where a subset $Y_0 = \{y_1, y_2\}$ of $Y$ is specified. The edge set of $FP(n, r)$ consists of all $r$-subsets of $[n]$ containing a member of the family $G = \{A \subset X : |A| = 3, x \in A, A \cap Y \neq \emptyset, A \cap Z \neq \emptyset\} \cup \{Y, Y_0 + x, Z + y_1, Z + y_2\}$.

By construction, $FP(n, r)$ is an intersecting $r$-graph with $\tau(FP(n, r)) = 3$. Frankl proved the following.

Theorem 5 (Frankl [8]). Let $r \geq 3$ and $n$ be sufficiently large. Then every intersecting $n$-vertex $r$-graph $H$ with $\tau(H) \geq 3$ satisfies $|H| \leq |FP(n, r)|$. Moreover, if $r \geq 4$, then equality is attained only if $H = FP(n, r)$.

He used the following folklore result.

Proposition 6. Every intersecting $3$-graph $H$ with $\tau(H) \geq 3$ satisfies $|H| \leq 10$.

Note that Erdős and Lovász [4] proved the more general result that for every $r \geq 2$ each intersecting $r$-graph $H$ with $\tau(H) = r$ has at most $r^r$ edges. But their proof gives the bound $25$ for $r = 3$, while Proposition 6 gives $10$.

In this short paper, we determine for large $n$, the structure of $H$ in the situations described above when $|H|$ is somewhat smaller than the bounds in Theorems 4 and 2. In particular, our Theorem 7 below answers for large $n$ the question of Han and Kohayakawa [14] at the end of their paper. We also use Theorem 5 to describe large dense hypergraphs $H$ with $\nu(H) \leq s$ and $\tau(H) = 2$. Related results can be found in [8][9].

2. Results

First we characterize the nontrivial intersecting $r$-graphs that have a bit fewer edges than $hm'(n, r)$. We need to describe three constructions before we can state our result.

Construction 5. For $r \geq 3$ and $t = n - r$, the $n$-vertex $r$-graph $HM(n, r, t)$ has $r$ distinct special vertices $x, x_1, \ldots, x_{r-1}$ and all edges $e$ such that
1) $\{x, x_i\} \subset e$ for any $i \in [r - 1]$, or
2) $\{x_1, \ldots, x_{r-1}\} \subset e$.

Similarly, for $r \geq 3$ and $1 \leq t \leq r - 1$, the $n$-vertex $r$-graph $HM(n, r, t)$ has $r + t$ distinct special vertices $x, x_1, \ldots, x_{r-1}, y_1, y_2, \ldots, y_t$ and all edges $e$ such that
1) $\{x, x_i\} \subset e$ for any $i \in [r - 1]$, or
2) $e = \{x_1, \ldots, x_{r-1}, y_j\}$ for all $1 \leq j \leq t$, and
3) $\{x, y_1, \ldots, y_t\} \subset e$.

Let $hm(n, r, t) = |HM(n, r, t)|$. Note that $HM(n, r, 1) = HM(n, r)$, and $HM(n, r, 2) = HM'(n, r)$. For $n$ large, we have the inequalities

$hm(n, r) = hm(n, r, 1) > \cdots > hm(n, r, r - 1) = hm(n, r, r) < hm(n, r, n - r)$.

Note that $HM(n, r, t)$ is intersecting, $\tau(HM(n, r, t)) = 2$, and $HM(n, r, t) \not\subset HM(n, r, t - 1)$. Also, for fixed $r \geq 4$ and $2 \leq t \leq n - r$,

$hm(n, r, t) \sim (r - 1) \binom{n}{r - 2}$. 
Theorem 7. The \( n \)-vertex \( r \)-graph \( HM(n, r, 0) \) has 3 special vertices \( x, x_1, x_2 \) and all edges that contain at least two of these 3 vertices.

By definition,

\[ |HM(n, r, 0)| = 3 \left( \frac{n-3}{r-2} \right) + \left( \frac{n-3}{r-3} \right). \]  

Construction 7. The \( n \)-vertex \( r \)-graph \( HM''(n, r) \) has \( r + 3 \) special vertices \( x, x_1, \ldots, x_{r-2} \) and \( y_1, y'_1, y_2, y'_2 \) and all edges \( e \) such that

1) \( \{x, x_i\} \subseteq e \) for some \( i \in [r-2] \), or
2) \( \{y_1, y_2\} \subseteq e \) or \( \{x, y_1, y_2\} \subseteq e \) or \( \{x, y'_1, y'_2\} \subseteq e \), or
3) \( e = \{x_1, \ldots, x_{r-2}, y_1, y'_1\} \), or \( e = \{x_1, \ldots, x_{r-2}, y_2, y'_2\} \).

Note that \( HM''(n, r) \) is intersecting, \( \tau(HM''(n, r)) = 2 \), and \( HM''(n, r) \not\subseteq HM(n, r, t) \) for any \( t \). Let \( hm''(n, r) = |HM''(n, r)| \) so that for \( r \geq 5 \),

\[ hm''(n, r) = \left( \frac{n-1}{r-1} \right) - \left( \frac{n-r+1}{r-1} \right) + 4 \left( \frac{n-r-3}{r-3} \right) \\
+ 4 \left( \frac{n-r-3}{r-4} \right) + \left( \frac{n-r-3}{r-5} \right) + 2 \\
\sim (r-2) \left( \frac{n}{r-2} \right). \]  

Theorem 7. Fix \( r \geq 4 \). Let \( n \) be sufficiently large. If \( H \) is an \( n \)-vertex intersecting \( r \)-graph with \( \tau(H) \geq 2 \) and \( |H| > hm''(n, r) \), then \( H \subseteq HM(n, r, t) \) for some \( t \in \{1, \ldots, r-1, n-r\} \) or \( r = 4 \) and \( H \subseteq HM(n, 4, 0) \). The bound on \( H \) is sharp due to \( HM''(n, r) \).

When \( r = 3 \) we are able to obtain stronger results than Theorem 7 and describe the structure of almost all intersecting 3-graphs. We will use the following construction.

Construction 8. Let \( n \geq 6 \).

- For \( i = 0, 1, 2 \), let \( H_i(n) = HM(n, 3, i) \) and \( H(n) = EM(n, 3, 1) \).

- The \( n \)-vertex 3-graph \( H_3(n) \) has special vertices \( v_1, v_2, y_1, y_2, y_3 \) and its edges are the \( n - 2 \) edges containing \( \{v_1, v_2\} \) and the 6 edges each of which contains one of \( v_1 \) and \( v_2 \) and two of \( y_1, y_2, y_3 \).

- Each of the \( n \)-vertex 3-graphs \( H_4(n) \) and \( H_5(n) \) has 6 special vertices \( v_1, v_2, z_{1,1}^{z_{1,1}}, z_{1,2}^{z_{1,2}}, z_{2,1}^{z_{2,1}}, z_{2,2}^{z_{2,2}} \) and contains all edges containing \( \{v_1, v_2\} \). Apart from these, \( H_4(n) \) contains edges

\[ v_1 z_{1,1}^{z_{1,1}}, v_1 z_{2,2}^{z_{2,2}}, v_2 z_{1,1}^{z_{1,1}}, v_2 z_{2,2}^{z_{2,2}}, v_2 z_{1,2}^{z_{1,2}}, v_2 z_{1,2}^{z_{1,2}}, v_2 z_{2,1}^{z_{2,1}}, v_2 z_{2,1}^{z_{2,1}} \]

and \( H_5(n) \) contains edges

\[ v_1 z_{1,1}^{z_{1,1}}, v_1 z_{2,2}^{z_{2,2}}, v_1 z_{1,2}^{z_{1,2}}, v_1 z_{2,1}^{z_{2,1}}, v_2 z_{1,1}^{z_{1,1}}, v_2 z_{1,1}^{z_{1,1}}, v_2 z_{1,2}^{z_{1,2}}, v_2 z_{1,2}^{z_{1,2}}, v_2 z_{2,1}^{z_{2,1}}, v_2 z_{2,1}^{z_{2,1}}. \]

Theorem 8. Let \( H \) be an intersecting 3-graph and \( n = |V(H)| \geq 6 \). If \( \tau(H) \leq 2 \), then \( H \) is contained in one of \( H(n), H_0(n), \ldots, H_5(n) \). This yields that

(a) if \( |H| \geq 11 \), then \( H \) is contained in one of \( H(n), H_0(n), \ldots, H_5(n) \);
(b) if \( |H| > n + 4 \), then \( H \) is contained in \( H(n), H_0(n), H_1(n) \) or \( H_2(n) \).
The restriction $|H| \geq 11$ cannot be weakened because of $K_5^3$ and $|H| > n + 4$ cannot be weakened because $|H_3(n)| = |H_4(n)| = |H_5(n)| = n + 4$.

To prove an analog of Theorem 8 for $r$-graphs, we need an extension of Construction 8.

**Construction 9.** Let $i \geq r + 1$. For $i = 0, \ldots, 5$, let the $r$-graph $H_i^r(n)$ have the vertex set of the $3$-graph $H_i(n)$ and the edge set of $H_i^r(n)$ consist of all $r$-tuples contained in an edge of $H_i(n)$.

By definition, $H_0^r(n) = HM(n, r, 0)$. Each $H_i^r(n)$ is intersecting, since each $H_i(n)$ is intersecting. Using Theorem 5 we extend Theorem 8 as follows:

**Theorem 9.** Let $r \geq 4$ be fixed and $n$ be sufficiently large. Then there is $C > 0$ such that for every intersecting $n$-vertex $r$-graph $H$ with $|H| > |FP(n, r)| = O(n^{r-3})$, one can delete from $H$ at most $Cn^{r-4}$ edges so that the resulting $r$-graph $H'$ is contained in one of $H_0^r(n), \ldots, H_5^r(n), EM(n, r, 1)$.

The results above naturally extend to $r$-graphs $H$ with $\nu(H) \leq s$. For example, Theorem 7 extends to the following result which implies Theorem 8 for large $n$.

**Theorem 10.** Fix $r \geq 4$ and $s \geq 1$. Let $n$ be sufficiently large. If $H$ is an $n$-vertex $r$-graph with $\nu(H) \leq s$ and $|H| > e n(n, r, s - 1) + hm''(n - s + 1, r)$, then $V(H)$ contains a subset $Z = \{z_1, \ldots, z_{s-1}\}$ such that either $\tau(H - Z) = 1$ or $H - Z \subseteq HM(n - s - 1, r, t)$ for some $t \in \{1, \ldots, r - 1, n - s + 1, 1 - r\}$ or $r = 4$ and $H - Z \subseteq HM(n + 1, 1, 4, 0)$. The bound on $|H|$ is sharp.

Theorems 4 and 9 can be extended in a similar way. We leave this to the reader.

### 3. Proof of Theorem 7

The main tool used in the proof is the Delta-system method developed by Frankl (see, e.g., [5]). Recall that a $k$-sunflower system $S$ is a collection of distinct sets $S_1, \ldots, S_k$ such that for every $1 \leq i < j \leq k$, we have $S_i \cap S_j = \bigcap_{\ell=1}^k S_{\ell}$. The common intersection of the $S_i$ is the core of $S$. We will use the following fundamental result of Erdős and Rado [5].

**Lemma 11** (Erdős-Rado Sunflower Lemma [5]). For every $k, r \geq 2$ there exists $f(k, r) < k^r r!$ such that the following holds: every $r$-graph $H$ with no $k$-sunflower satisfies $|H| < f(k, r)$.

**Proof of Theorem 7.** Let $r \geq 4$ and $H$ be an $n$-vertex intersecting $r$-graph with $\tau(H) \geq 2$ and $|H| > hm''(n, r)$. Define $B^*(H)$ to be the set of $T \subseteq V(H)$ such that

(i) $0 < |T| < r$; and

(ii) $T$ is the core of an $(r + 1)|T|$-sunflower in $H$.

Define $B'(H) = \{T \in B^*(H) : \exists U \in B^*(H), U \subseteq T\}$ to be the set of all inclusion minimal elements in $B^*(H)$. Next, let $B''(H) = \{e \in H : \exists T \subseteq e, T \in B^*(H)\}$ be the set of edges in $H$ that contains no member of $B^*(H)$. Finally, set $B(H) = B'(H) \cup B''(H)$.

Let $B_i$ be the family of the sets in $B(H)$ of size $i$. Note that $B_0 = \emptyset$ for otherwise we have an $(r + 1)$-sunflower with core of size one and since $H$ is intersecting, this
forces \( H \) to be trivial. Similarly, if \( 2 \leq i \leq r - 1 \) and \( T, T' \in B_i \), then \( T \cap T' \neq \emptyset \), since otherwise \( H \) would have disjoint edges \( A \supset T \) and \( A' \supset T' \). Thus for each \( 2 \leq i \leq r - 1 \), \( B_i \) is an intersecting family. The following crucial claim proved by Frankl can be found in Lemma 1 in [6,8].

\[ \square \]

**Claim.** \( B_i \) contains no \((r + 1)^{i-1}\)-sunflower.

**Proof of Claim.** Suppose for contradiction that \( S_1, \ldots, S_{(r+1)^{i-1}} \) is an \((r + 1)^{i-1}\)-sunflower in \( B_i \) with core \( K \). By definition of \( B_i \), there is an \((r + 1)^{i-1}\)-sunflower \( S_1 = S_{1,1}, \ldots, S_{1,(r+1)^i} \) in \( H \) with core \( S_1 \). Since \( \sum S_2 \cup \cdots \cup S_{(r+1)^{i-1}} < (r + 1)(r + 1)^i \), and \( S_1 \) is an \((r + 1)^i\)-sunflower, there is a \( k = k(1) \) such that

\[ (S_1,k(1) - S_1) \cap (S_2 \cup S_3 \cup \cdots \cup S_{(r+1)^{i-1}}) = \emptyset. \]

Next, we use the same argument to define \( S_{2,k(2)} \) such that \( S_{2,k(2)} - S_2 \) is disjoint from \( S_{1,k(1)} \cup S_3 \cup \cdots \cup S_{(r+1)^{i-1}} \) and then \( S_{3,k(3)} \) such that \( S_{3,k(3)} - S_3 \) is disjoint from \( S_{1,k(1)} \cup S_{2,k(2)} \cup S_3 \cup \cdots \cup S_{(r+1)^{i-1}} \) and so on. Continuing in this way we finally obtain edges \( S_j,k(j) \) of \( H \) for all \( 1 \leq j \leq (r + 1)^{i-1} \) that form an \((r + 1)^{i-1}\)-sunflower with core \( K \). This implies that \( K \neq \emptyset \) as \( H \) is intersecting. Since \( |K| \leq i - 1 \), there exists a nonempty \( K' \subseteq K \) such that \( K' \subseteq B(H) \). But \( K' \subseteq S_j \) for all \( j \), so this contradicts the fact that \( S_j \in B(H) \).

Applying the Claim and Lemma \[11] yields \( |B_i| < f(\sum (r + 1)^{(r-1)^i}) \) for all \( i > 1 \). Every edge of \( H \) contains an element of \( B(H) \) so we can count edges of \( H \) by the sets in \( B(H) \). So for \( q = |B_2| \) we have

\[
hm''(n, r) < |H| \leq \sum_{B \in B_2} \binom{n - 2}{r - 2} + \sum_{i=3}^{r} \sum_{B \in B_i} \binom{n - i}{r - i} < q \binom{n - 2}{r - 2} + (r - 2) f((r + 1)^{i-1}, r) \binom{n}{r - 3}.
\]

Since \( hm''(n, r) \sim (r - 2) \binom{n - 2}{r - 2} \), this gives \( q \geq r - 2 \). On the other hand, \( B_2 \) is intersecting and thus the pairs in \( B_2 \) form either the star \( K_{1,q} \) or a \( K_3 \).

**Case 1.** \( B_2 \) is a \( K_3 \). Then to keep \( H \) intersecting, \( H \subseteq HM(n, r, 0) \). If \( r \geq 5 \), then by \[3\] and \[4\], \( |HM(n, r, 0)| < hm''(n, r) < |H| \), a contradiction. Thus \( r = 4 \) and \( H \subseteq HM(n, 4, 0) \), as claimed.

Since Case 1 is proved, we may assume that \( B_2 \) is a star with center \( x \) and the set of leaves \( X = \{x_1, \ldots, x_t\} \).

**Case 2** \( (q \geq r - 1) \). If \( q \geq r \), then \( q = r \) and since \( H \) is nontrivial, \( H \subseteq HM(n, r) \) and we are done. We may therefore assume that \( q = r - 1 \). Since \( \tau(H) \geq 2 \), there exists \( e \) such that \( x \notin e \in H \), and since \( H \) is intersecting we may assume that \( e = e_1 = X \cup \{y_1\} \). We may also assume that all edges of \( H \) that omit \( x \) are of the form \( e_i = X \cup \{y_i\} \), where \( 1 \leq i \leq t \). If \( t = 1 \), then \( H \subseteq HM(n, r) \) and we are done, so assume that \( t \geq 2 \). Any edge of \( H \) containing \( x \) that omits \( X \) must contain all \( \{y_1, \ldots, y_t\} \). Consequently, \( H \subseteq HM(n, r, t) \) for some \( t \in \{1, \ldots, r - 1, n - r\} \).

**Case 3** \( (q = r - 2) \). Let \( F_0 \) be the set of edges in \( H \) that contain \( x \) and intersect \( X \), \( F_1 \) be the set of edges of \( H \) disjoint from \( X \) and \( F_2 \) be the set of edges disjoint from
3.2 \((G, H) \subseteq (X, \nu)
\)

Case 3.1 \((\tau(G) = 1)\). Then \(G = K_{1,s}\) for some \(1 \leq s \leq n - r\). Let the partite sets of \(G\) be \(x_{r-1}\) and \(Y\). Then every edge in \(F_1\) must contain either \(x_{r-1}\) or \(Y\). Thus \(H \subseteq HM(n, r, t)\) for some \(t \in \{1, \ldots, r-1, n-r\}\), as claimed.

Case 3.2 \((\tau(G) \geq 2 \text{ and } \nu(G) = 1)\). Then \(G = K_3\) and every edge in \(F_1\) must contain at least two vertices of \(G\). Then \(|F_1| < 3{n-r-1 \choose r-3} \sim 3{r-3 \choose r-3}\) and thus \(|F_1 \cup F_2| = |F_1| + 3 \sim 3{n-r \choose r-3}\), contradicting (5).

Case 3.3 \((\nu(G) \geq 3)\). Let \(f_1, f_2, f_3\) be disjoint edges in \(G\). Then each edge in \(F_1\) has at least 4 vertices in \(f_1 \cup f_2 \cup f_3 \cup \{x\}\) and thus \(|F_1| = O(n^{r-4})\). If \(F_1 = \emptyset\), then \(H \subseteq HM(n, r, n-r)\), as claimed. Suppose there is \(e_0 \in F_1\). Then each \(f \in G\) meets \(e_0 - x\) and thus \(|G| = |F_2| \leq (r-1)(n-2r+2) + \eta^{-1}\). Thus if \(r \geq 5\), then \(|F_1| \leq O(n^{r-3}) + O(n) = o(n^{r-3})\), contradicting (5). Moreover, if \(r = 4\), then \(|F_2| \leq 3(n-6) + 3 |F_1 \cup F_2| \leq O(n^{r-3} + 3n < 4{n \choose 1}\), again contradicting (5).

Case 3.4 \((\nu(G) = 2)\). Say that a vertex \(v\) is big if \(d_G(v) \geq 2r\). Let \(v_1, \ldots, v_s\) be all the big vertices in \(G\). Since \(\nu(G) = 2\), \(s \leq 2\). Since \(H\) is intersecting,

\(6\)

Every edge in \(F_1\) contains all big vertices.

Suppose first, \(s = 2\). Then to have \(\nu(G) = 2\), all edges in \(F_2\) are incident with \(v_1\) or \(v_2\); thus \(|F_2| < 2n\). On the other hand, in this case by (5), \(|F_1| \leq {n-r-1 \choose r-3}\). Together, this contradicts (5).

Suppose now, \(s = 1\). Then to have \(\nu(G) = 2\), we need \(|F_2| \leq d_G(v_1) + 2r \leq n + 2r\). On the other hand, \(\nu(G) = 2\), \(G\) has an edge \(v'v''\) disjoint from \(v_1\). It follows that each edge in \(F_1\) meets \(v'v''\). By this and (5), \(|F_1| \leq 2{n-r \choose r-3}\) and thus \(|F_1 \cup F_2| \leq n + 2r + 2{n-r \choose r-3}\), contradicting (5).

Finally, suppose \(s = 0\). Let edges \(y_1y'_1\) and \(y_2y'_2\) form a matching in \(G\). If \(G\) has no other edges, then \(H\) is contained in \(HM''(n, r)\). So there is a third edge in \(G\). Still, since \(\nu(G) = 2\), each edge of \(G\) is incident with \(\{y_1, y_1', y_2, y_2'\}\) which by \(s = 0\) yields \(|F_2| = |G| < 8r\). If an edge in \(G\) is \(y_1y_3\), then each edge in \(F_1\) contains \(\{y_1, y_2\}\) or \(\{y_1, y_2'\}\) or \(\{y_1', y_2, y_3\}\) or \(\{y_1, y_2', y_3\}\); thus \(|F_1| \leq 2{n-r \choose r-3} + 2{n-r \choose n-4} \sim 2{n-r \choose r-3}\). This together with \(|F_2| \leq 8r\) contradicts (5). If this third edge is \(y_1y_2\), then we get a similar contradiction.

\[\square\]

4. On 3-graphs

**Lemma 12.** Let \(n \geq 6\) and \(H\) be an intersecting 3-graph. If \(H\) has a vertex \(x\) such that \(H - x\) has at most two edges, then \(H\) is contained in one of \(H(n), H_0(n), H_1(n), H_2(n), H_4(n)\).

**Proof.** If \(H - x\) has no edges, then \(H \subseteq H(n)\), and if \(H - x\) has one edge, then \(H \subseteq H_1(n)\). Suppose \(H - x\) has two edges, \(e_1\) and \(e_2\). If \(|e_1 \cap e_2| = 2\), then we may
assume \( e_1 = \{x_1, x_2, y_1\} \) and \( e_2 = \{x_1, x_2, y_2\} \). In this case, each edge in \( H - e_1 - e_2 \) contains \( x \) and either intersects \( \{x_1, x_2\} \) or coincides with \( \{x, y_1, y_2\} \). This means \( H \subseteq H_5(n) \).

If \( |e_1 \cap e_2| = 1 \), then we may assume \( e_1 = \{y, v_1, w_1\} \) and \( e_2 = \{y, v_2, w_2\} \). In this case, each edge in \( H - e_1 - e_2 \) contains \( x \) and either contains \( y \) or intersects each of \( \{v_1, w_1\} \) and \( \{v_2, w_2\} \). This means \( H \subseteq H_4(n) \).

**Proof of of Theorem 8.** Let \( n \geq 6 \) and \( H \) be an \( n \)-vertex intersecting 3-graph with \( \tau(H) \leq 2 \) not contained in any of \( H(n), H_0(n), \ldots, H_5(n) \). Write \( H_i \) for \( H_i(n) \).

If \( \tau(H) = 1 \), then \( H \subseteq H(n) \). So, suppose a set \( \{v_1, v_2\} \) covers all edges of \( H \), but \( H \) is not a star. Let \( E_0 = \{e \in H : \{v_1, v_2\} \subseteq e\} \), and for \( i = 1, 2 \), let \( E_i = \{e \in H : v_{3-i} \not\in e\} \).

By Lemma 12, \( |E_1|, |E_2| \geq 3 \). For \( i = 1, 2 \), let \( F_i \) be the subgraph of the link graph of \( v_i \) formed by the edges in \( E_i \). If \( \tau(F_i) \geq 3 \), then any edge \( e \in E_{3-i} \) does not cover some edge \( f \in F_i \) and thus is disjoint from \( f + v_1 \in H \), a contradiction. Thus \( \tau(F_1) \leq 2 \) and \( \tau(F_2) \leq 2 \).

**Case 1** (\( \tau(F_1) = 1 \)). Suppose \( x_1 \) is a dominating vertex in \( F_1 \). Since \( |F_1| = |E_1| \geq 3 \), \( x_1 \) is the dominating vertex in \( F_1 \) and we may assume that \( x_1 x_2, x_1 x_3, x_1 x_4 \in F_1 \). But to cover these 3 edges, each edge in \( F_2 \) must contain \( x_1 \). Thus \( H \subseteq H_0(n) \), as claimed.

**Case 2** (\( \tau(F_1) = \tau(F_2) = 2 \)). If say \( F_1 \) contains a triangle \( T = y_1 y_2 y_3 \), then \( F_2 \) cannot contain an edge not in \( T \) and thus \( F_2 = T \) and by symmetry \( F_1 = T \). Thus \( H \) is contained in \( H_4 \).

So the remaining case is that each of \( F_i \) contains a matching \( M_i = \{z_{1,i}, z'_{1,i}, z_{2,i}, z'_{2,i}\} \).

Since each edge of \( F_1 \) intersects each edge of \( F_2 \), we may assume \( z_{1,2} = z_{1,1}, z'_{1,2} = z_{2,1}, z'_{2,2} = z_{1,1} \), \( z'_{2,2} = z_{2,1} \). The only other edges that may have \( F_2 \) are \( f_1 = z_{1,1}, z_{2,1} \) and \( f_2 = z'_{1,1}, z_{2,1} \). Since \( |F_2| \geq 3 \), we may assume \( f_1 \in F_2 \). Then the only third edge that \( F_1 \) may contain is also \( f_1 \). It follows that \( H \) is contained in \( H_5 \). This proves the main part of the theorem.

To prove part (a), assume \( H \) is an intersecting \( n \)-vertex 3-graph with \( |H| \geq 11 \). Since \( |K_6^3| = 10 < |H|, n \geq 6 \). By Proposition 8, \( \tau(H) \leq 2 \). So part (a) is implied by the main claim of the theorem. Part (b) follows from the fact that each of \( H_3, H_4, H_5 \) has \( n + 4 \) edges.

**5. PROOF OF THEOREM 9**

Let \( H \) be as in the statement. By Theorem 8, \( \tau(H) \leq 2 \). So, suppose a set \( \{v_1, v_2\} \) covers all edges of \( H \). Let \( E_0 = \{e \in H : \{v_1, v_2\} \subseteq e\} \), and for \( i = 1, 2 \), let \( E_i = \{e \in H : v_{3-i} \not\in e\} \).

For \( E_1 \cup E_2 \), construct the family \( B(H) = B_1 \cup B_2 \cup \ldots B_r \) as in the previous proofs. Recall that by the minimality of the sets in \( B_i \),

\[
X \not\subseteq Y \quad \text{for all distinct} \quad X, Y \in B(H),
\]

and since \( H \) is intersecting,

\[
B(H) \text{ is intersecting.}
\]
If \( B_1 \neq \emptyset \), say \( \{v_0\} \in B_1 \), then by (7) and (8), and \( B(H) = \{\{v_0\}\} \). This means either \( H \subseteq H(n, r) \) (when \( v_0 \in \{v_1, v_2\} \)), or \( H \subseteq H_5(n) \) (when \( v_0 \notin \{v_1, v_2\} \)), and the theorem holds. So, let \( B_1 = \emptyset \).

Let \( H' \) be obtained from \( H \) by deleting all edges not containing a member of \( B' = B_2 \cup B_3 \). Then \( |H - H'| \leq Cn^{r-4} \). Since \( \{v_1, v_2\} \) dominates \( H \),

\[
\text{(9) each } D \in B' \text{ must contain either } v_1 \text{ or } v_2.
\]

For \( i = 1, 2 \), let \( B'_i \) be the set of the members of \( B' \) containing \( v_i \).

Define the auxiliary 3-graph \( H'' \) with vertex set \( V(H) \) as follows. The edges of \( H'' \) are all members of \( B_3 \) and each triple \( f \) that contains a member of \( B_2 \) and is contained in an \( e \in H' \).

By (8), \( H'' \) is intersecting. By (7), \( \tau(H'') \leq 2 \). If \( \tau(H'') = 1 \), then \( H' \) is a star. Suppose \( \tau(H'') = 2 \). By Theorem 11, \( H'' \) is contained in one of \( H(n), H_0(n), \ldots, H_5(n) \). But then \( H' \) is contained in one of \( H'_0(n), \ldots, H'_5(n), EM(n, r, 1) \), as claimed.

\[
\text{□}
\]

6. **Proof of Theorem 10**

Recall that \( r \geq 4, s \geq 1, n \) is sufficiently large and \( H \) is an \( n \)-vertex \( r \)-graph with \( \nu(H) \leq s \) and \( |H| > em(n, r, s - 1) + hm''(n - s + 1, r) \). We are to show that \( V(H) \) contains a subset \( Z = \{z_1, \ldots, z_{s-1}\} \) such that either \( \tau(H - Z) = 1 \) or \( H - Z \leq HM(n - s + 1, r, t) \) for some \( t \in \{1, \ldots, r - 1, n - s + 1 - r\} \) or \( r = 4 \) and \( H - Z \leq HM(n - s + 1, 4,0) \).

Define \( B(H) \) and \( B_1 \) as in the previous proofs with the slight change that \( T \in B(H) \) lies in an \((rs)^{|T|} + 1\)-sunflower (instead of an \((r + 1)^{|T|}\)-sunflower). Then the following claim holds (with an identical proof).

**Claim.** \( B_i \) contains no \((rs)^i\)-sunflower.

Using the Claim and Lemma 11, we obtain \( |B_i| < f((rs)^i, i) \) for all \( 1 \leq i \leq r \). As before, setting \( h = |B_1| \) we have

\[
|H| \leq \sum_{B \in B_1} \left( \frac{n - 1}{r - 1} \right) + \sum_{i=2}^{r} \sum_{B \in B_i} \left( \frac{n - i}{r - i} \right) < h \left( \frac{n - 1}{r - 1} \right) + (r - 1)f((rs)^r, r) \left( \frac{n}{r - 2} \right).
\]

Since \( |H| > em(n, r, s - 1) + hm''(n - s + 1, r) \sim s \left( \frac{n}{r - 1} \right) \) and \( n \) is large, this immediately gives \( h \geq s - 1 \). Consider distinct vertices \( z_1, \ldots, z_{s-1} \in B_1 \) and the set of edges \( F \subset H \) omitting \( z_1, \ldots, z_{s-1} \). If \( F \) is not intersecting, then let \( e, e' \) be two disjoint edges in \( F \). There exists a matching \( e_1, \ldots, e_{s-1} \in H \) with \( z_i \in e_i \) and \( (e \cup e') \cap e_i = \emptyset \) for all \( 1 \leq i \leq s - 1 \). Note that we can produce the \( e_i \) one by one since each \( z_i \) forms the core of an \((rs)^2\)-sunflower in \( H \) due to the definition of \( B_1 \). We obtain the matching \( e, e', e_1, \ldots, e_{s-1} \) contradicting \( \nu(H) \leq s \). Consequently, we may assume that \( F \) is intersecting. Because \( |H| > em(n, r, s - 1) + hm''(n - s + 1, r) \) we have \( |F| > hm''(n - s + 1, r) \). Now we apply Theorem 11 to \( F \) to conclude that Theorem 10 holds.

\[
\text{□}
\]

7. **Concluding remarks**

Say that a hypergraph \( H \) is \( t \)-irreducible, if \( \nu(H) = t \) and \( \nu(H - x) = t \) for every \( x \in V(H) \). Frankl [10] presented a family of \( n \)-vertex \( t \)-irreducible \( r \)-graphs.
$PF(n, r, t)$ such that

$$pf(n, r, t) = |PF(n, r, t)| \sim r \left(\frac{t-1}{2}\right) \binom{n}{r-2}.$$ 

He also proved

**Theorem 13** ([10]). Let $r \geq 4$, $t \geq 1$, and let $n$ be sufficiently large. Then every $n$-vertex $t$-irreducible $r$-graph $H$ has at most $pf(n, r, t)$ edges with equality only if $H = PF(n, r, t)$.

Using this result, one can prove the following.

**Lemma 14.** For every $r \geq 3$, $s \geq t \geq 2$, if $n$ is large, and $H$ is an $n$-vertex $r$-graph with $\nu(H) = s$ and

$$|H| > em(n, r, s - t) + pf(n - s + t, r, t),$$

then there exists $X \subseteq V(H)$ with $|X| = s - t + 1$ such that $\nu(H - X) = t - 1$. The bound on $|H|$ is sharp.

This in turn implies the following claim.

**Theorem 15.** For every $r \geq 3$ and $s \geq 2$ there exists $c > 0$ such that the following holds. If $n$ is large, and $H$ is an $n$-vertex $r$-graph with $\nu(H) = s$ and

$$|H| > em(n, r, s - 2) + pf(n - s + 2, r, 2),$$

then either

1) there exists $H' \subset H$ with $|H'| < cn^{r-3}$ and $\tau(H - H') \leq s$ or
2) there exist an $X \subset V(H)$ with $|X| = s - 1$ and $u, v, w \in V(H - X)$ such that every edge of $H - X$ contains at least two elements of $\{u, v, w\}$.

We leave the details of the proofs to the reader.

Most of the proofs in this paper are rather simple applications of the early version of the Delta-system method. There has been renewed interest in stability versions for problems in extremal set theory, so the general message of this work is that the Delta-system method can quickly give some structural information about problems in extremal set theory, a fact that was already shown in several papers by Frankl and Füredi in the 1980s. For more advanced recent applications of the Delta-system method, see the papers of Füredi [12] and Füredi-Jiang [13].

**ACKNOWLEDGMENTS**

The authors thank Peter Frankl for helpful comments on an earlier version of the paper and Jozsef Balogh and Shagnik Das for attracting our attention to [14]. We also thank a referee for helpful comments.

**REFERENCES**


THE STRUCTURE OF LARGE INTERSECTING FAMILIES


University of Illinois at Urbana-Champaign, Urbana, Illinois 61801 — and — Sobolev Institute of Mathematics, Novosibirsk 630090, Russia

E-mail address: kostochk@math.uiuc.edu

Department of Mathematics, Statistics, and Computer Science, University of Illinois at Chicago, Chicago, Illinois 60607

E-mail address: mubayi@uic.edu

Licensed to Univ of Ill at Urbana-Champaign. Prepared on Tue Dec 12 15:03:40 EST 2017 for download from IP 130.126.111.111.
License or copyright restrictions may apply to redistribution; see http://www.ams.org/journal-terms-of-use