DP-colorings of graphs with high chromatic number
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A B S T R A C T
DP-coloring is a generalization of list coloring introduced recently by Dvořák and Postle (2015). We prove that for every n-vertex graph G whose chromatic number \( \chi(G) \) is “close” to \( n \), the DP-chromatic number of \( G \) equals \( \chi(G) \). “Close” here means \( \chi(G) \geq n - O(\sqrt{n}) \), and we also show that this lower bound is best possible (up to the constant factor in front of \( \sqrt{n} \)), in contrast to the case of list coloring.

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1. Introduction

We use standard notation. In particular, \( \mathbb{N} \) denotes the set of all nonnegative integers. For a set \( S \), \( \text{Pow}(S) \) denotes the power set of \( S \), i.e., the set of all subsets of \( S \). All graphs considered here are finite, undirected, and simple. For a graph \( G \), \( V(G) \) and \( E(G) \) denote the vertex and the edge sets of \( G \), respectively. For a set \( U \subseteq V(G) \), \( G[U] \) is the subgraph of \( G \) induced by \( U \). Let \( G - U := G[V(G) \setminus U] \), and for \( u \in V(G) \), let \( G - u := G \setminus \{u\} \). For \( U_1, U_2 \subseteq V(G) \), let \( E_G(U_1, U_2) \subseteq E(G) \) denote the set of all edges in \( G \) with one endpoint in \( U_1 \) and the other one in \( U_2 \). For \( u \in V(G) \), \( N_G(u) \subseteq V(G) \) denotes the set of all neighbors of \( u \), and \( \deg_G(u) := |N_G(u)| \) is the degree of \( u \) in \( G \). We use \( \delta(G) \) to denote the minimum degree of \( G \), i.e., \( \delta(G) := \min_{u \in V(G)} \deg_G(u) \). For \( U \subseteq V(G) \), let \( N_G(U) := \bigcup_{u \in U} N_G(u) \). To simplify notation, we write \( N_G(u_1, \ldots, u_k) \) instead of \( N_G(\{u_1, \ldots, u_k\}) \). A set \( I \subseteq V(G) \) is independent if \( I \cap N_G(I) = \emptyset \), i.e., if \( uv \notin E(G) \) for all \( u, v \in I \). We denote the family of all independent sets in a graph \( G \) by \( \mathcal{I}(G) \). The complete \( k \)-vertex graph is denoted by \( K_k \).

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1.1. The Noel–Reed–Wu theorem for list coloring

Recall that a proper coloring of a graph $G$ is a function $f : V(G) \rightarrow Y$, where $Y$ is a set of colors, such that $f(u) \neq f(v)$ for every edge $uv \in E(G)$. The smallest $k \in \mathbb{N}$ such that there exists a proper coloring $f : V(G) \rightarrow Y$ with $|Y| = k$ is called the chromatic number of $G$ and is denoted by $\chi(G)$.

List coloring was introduced independently by Vizing [10] and Erdős, Rubin, and Taylor [5]. A list assignment for a graph $G$ is a function $L : V(G) \rightarrow \text{Pow}(Y)$, where $Y$ is a set. For each $u \in V(G)$, the set $L(u)$ is called the list of $u$, and its elements are the colors available for $u$. A proper coloring $f : V(G) \rightarrow Y$ is called an L-coloring if $f(u) \in L(u)$ for each $u \in V(G)$. The list chromatic number $\chi_L(G)$ of $G$ is the smallest $k \in \mathbb{N}$ such that $G$ is $L$-colorable for each list assignment $L$ with $|L(u)| \geq k$ for all $u \in V(G)$. It is an immediate consequence of the definition that $\chi_L(G) \geq \chi(G)$ for every graph $G$.

It is well-known (see, e.g., [5, 10]) that the list chromatic number of a graph can significantly exceed its ordinary chromatic number. Moreover, there exist 2-colorable graphs with arbitrarily large list chromatic numbers. On the other hand, Noel, Reed, and Wu [6] established the following result, which was conjectured by Ohba [7, Conjecture 1.3]:

**Theorem 1.1 (Noel–Reed–Wu [6]).** Let $G$ be an $n$-vertex graph with $\chi(G) \geq (n-1)/2$. Then $\chi_L(G) = \chi(G)$.

The following construction was first studied by Ohba [7] and Enomoto, Ohba, Ota, and Sakamoto [4]. For a graph $G$ and $s \in \mathbb{N}$, let $J(G, s)$ denote the join of $G$ and a copy of $K_s$, i.e., the graph obtained from $G$ by adding $s$ new vertices that are adjacent to every vertex in $V(G)$ and to each other. It is clear from the definition that for all $G$ and $s$, $\chi(J(G, s)) = \chi(G) + s$. Moreover, we have $\chi_L(J(G, s)) \leq \chi_L(G) + s$; however, this inequality can be strict. Indeed, Theorem 1.1 implies that for every graph $G$ and every $s \geq |V(G)| - 2\chi(G) - 1$,

$$\chi_L(J(G, s)) = \chi(J(G, s)),$$

even if $\chi_L(G)$ is much larger than $\chi(G)$.

In view of this observation, it is interesting to consider the following parameter:

$$Z_L(G) := \min\{s \in \mathbb{N} : \chi_L(J(G, s)) = \chi(J(G, s))\},$$

(1.1)
i.e., the smallest $s \in \mathbb{N}$ such that the list and the ordinary chromatic numbers of $J(G, s)$ coincide. The parameter $Z_L(G)$ was explicitly defined by Enomoto, Ohba, Ota, and Sakamoto in [4, page 65] (they denoted it $\psi(G)$). Recently, Kim, Park, and Zhu (personal communication, 2016) obtained new lower bounds on $Z_L(K_{2,n}), Z_L(K_{n,n}),$ and $Z_L(K_{n,n,n})$. One can also consider, for $n \in \mathbb{N},$

$$Z_L(n) := \max\{Z_L(G) : |V(G)| = n\}.$$  
(1.2)

The parameter $Z_L(n)$ is closely related to the Noel–Reed–Wu Theorem, since, by definition, there exists a graph $G$ on $n + Z_L(n) - 1$ vertices whose ordinary chromatic number is at least $Z_L(n)$ and whose list and ordinary chromatic numbers are distinct. The finiteness of $Z_L(n)$ for all $n \in \mathbb{N}$ was first established by Ohba [7, Theorem 1.3]. Theorem 1.1 yields an upper bound $Z_L(n) \leq n - 5$ for all $n \geq 5$; on the other hand, a result of Enomoto, Ohba, Ota, and Sakamoto [4, Proposition 6] implies that $Z_L(n) \geq n - O(\sqrt{n})$.

1.2. DP-colorings and the results of this paper

The goal of this note is to study analogs of $Z_L(G)$ and $Z_L(n)$ for the generalization of list coloring that was recently introduced by Dvořák and Postle [3], which we call DP-coloring. Dvořák and Postle invented DP-coloring to attack an open problem on list coloring of planar graphs with no cycles of certain lengths.

**Definition 1.2.** Let $G$ be a graph. A cover of $G$ is a pair $(L, H)$, where $H$ is a graph and $L : V(G) \rightarrow \text{Pow}(V(H))$ is a function, with the following properties:

- the sets $L(u), u \in V(G)$, form a partition of $V(H)$;
- if $u, v \in V(G)$ and $L(u) \cap N_H(L(u)) \neq \emptyset$, then $v \in \{u\} \cup N_G(u);$. 


- each of the graphs $H[L(u)], u \in V(G)$, is complete;
- if $uv \in E(G)$, then $E_H[L(u), L(v)]$ is a matching (not necessarily perfect and possibly empty).

**Definition 1.3.** Let $G$ be a graph and let $(L, H)$ be a cover of $G$. An $(L, H)$-coloring of $G$ is an independent set $I \subseteq \mathcal{I}(H)$ of size $|V(G)|$. Equivalently, $I \subseteq \mathcal{I}(H)$ is an $(L, H)$-coloring of $G$ if $|I \cap L(u)| = 1$ for all $u \in V(G)$.

**Remark 1.4.** Suppose that $G$ is a graph, $(L, H)$ is a cover of $G$, and $G'$ is a subgraph of $G$. In such situations, we will allow a slight abuse of terminology and speak of $(L, H)$-colorings of $G'$ (even though, strictly speaking, $(L, H)$ is not a cover of $G'$).

The **$DP$-chromatic number** $\chi_{DP}(G)$ of a graph $G$ is the smallest $k \in \mathbb{N}$ such that $G$ is $(L, H)$-colorable for each cover $(L, H)$ with $|I_L(u)| \geq k$ for all $u \in V(G)$.

To show that $DP$-colorings indeed generalize list colorings, consider a graph $G$ and a list assignment $L$ for $G$. Define a graph $H$ as follows: Let $V(H) := \{(u, c) : u \in V(G) \text{ and } c \in L(u)\}$ and let 

$$(u_1, c_1)(u_2, c_2) \in E(H) :\iff (u_1 = u_2 \text{ and } c_1 \neq c_2) \text{ or } (u_1u_2 \in E(G) \text{ and } c_1 = c_2).$$

For $u \in V(G)$, let $\hat{L}(u) := \{(u, c) : c \in L(u)\}$. Then $(\hat{L}, H)$ is a cover of $G$, and there is a one-to-one correspondence between $L$-colorings and $(\hat{L}, H)$-colorings of $G$. Indeed, if $f$ is an $L$-coloring of $G$, then the set $I_f := \{(u, f(u)) : u \in V(G)\}$ is an $(\hat{L}, H)$-coloring of $G$. Conversely, given an $(\hat{L}, H)$-coloring $I$ of $G$, we can define an $L$-coloring $f_I$ of $G$ by the property $(u, f_I(u)) \in I$ for all $u \in V(G)$. Thus, list colorings form a subclass of $DP$-colorings. In particular, $\chi_{DP}(G) \geq \chi_L(G)$ for each graph $G$.

Some upper bounds on list-chromatic numbers hold for $DP$-chromatic numbers as well. For example, $\chi_{DP}(G) \leq d + 1$ for any $d$-degenerate graph $G$. Dvořák and Postle [3] pointed out that Thomassen’s bounds [8,9] on the list chromatic numbers of planar graphs hold also for their $DP$-chromatic numbers; in particular, $\chi_{DP}(G) \leq 5$ for every planar graph $G$. On the other hand, there are also some striking differences between $DP$- and list coloring. For instance, even cycles are $2$-list-colorable, while their $DP$-chromatic number is $3$; in particular, the orientation theorems of Alon–Tarsi [1] and the Bondy–Boppana–Siegel Lemma (see [1]) do not extend to $DP$-coloring (see [2] for further examples of differences between list and $DP$-coloring).

By analogy with (1.1) and (1.2), we consider the parameters

$$Z_{DP}(G) := \min\{s \in \mathbb{N} : \chi_{DP}(\mathbf{J}(G, s)) = \chi(\mathbf{J}(G, s))\},$$

and

$$Z_{DP}(n) := \max\{Z_{DP}(G) : |V(G)| = n\}.$$

Our main result asserts that for every graph $G$, $Z_{DP}(G)$ is finite:

**Theorem 1.5.** Let $G$ be a graph with $n$ vertices, $m$ edges, and chromatic number $k$. Then $Z_{DP}(G) \leq 3m$. Moreover, if $\delta(G) \geq k - 1$, then

$$Z_{DP}(G) \leq 3m - \frac{3}{2}(k - 1)n.$$

**Corollary 1.6.** For all $n \in \mathbb{N}$, $Z_{DP}(n) \leq 3n^2/2$.

Note that the upper bound on $Z_{DP}(n)$ given by **Corollary 1.6** is quadratic in $n$, in contrast to the linear upper bound on $Z_L(n)$ implied by **Theorem 1.1**. Our second result shows that the order of magnitude of $Z_{DP}(n)$ is indeed quadratic:

**Theorem 1.7.** For all $n \in \mathbb{N}$, $Z_{DP}(n) \geq n^2/4 - O(n)$.

**Corollary 1.6** and **Theorem 1.7** also yield the following analog of **Theorem 1.1** for $DP$-coloring:

**Corollary 1.8.** For $n \in \mathbb{N}$, let $r(n)$ denote the minimum $r \in \mathbb{N}$ such that for every $n$-vertex graph $G$ with $\chi(G) \geq r$, we have $\chi_{DP}(G) = \chi(G)$. Then

$$n - r(n) = \Theta(\sqrt{n}).$$
We prove Theorem 1.5 in Section 2 and Theorem 1.7 in Section 3. The derivation of Corollary 1.8 from Corollary 1.6 and Theorem 1.7 is straightforward; for completeness, we include it at the end of Section 3.

2. Proof of Theorem 1.5

For a graph $G$ and a finite set $A$ disjoint from $V(G)$, let $J(G, A)$ denote the graph with vertex set $V(G) \cup A$ obtained from $G$ by adding all edges with at least one endpoint in $A$ (i.e., $J(G, A)$ is a concrete representative of the isomorphism type of $(G, |A|)$).

First we prove the following more technical version of Theorem 1.5:

**Theorem 2.1.** Let $G$ be a $k$-colorable graph. Let $A$ be a finite set disjoint from $V(G)$ and let $(L, H)$ be a cover of $J(G, A)$ such that for all $a \in A$, $|L(a)| \geq |A| + k$. Suppose that

\[
|A| \geq \frac{3}{2} \sum_{v \in V(G)} \max\{\deg_G(v) + |A| - |L(v)| + 1, 0\}.
\]

Then $J(G, A)$ is $(L, H)$-colorable.

**Proof.** For a graph $G$, a set $A$ disjoint from $V(G)$, a cover $(L, H)$ of $J(G, A)$, and a vertex $v \in V(G)$, let

\[
\sigma(G, A, L, H, v) := \max\{\deg_G(v) + |A| - |L(v)| + 1, 0\}
\]

and

\[
\sigma(G, A, L, H) := \sum_{v \in V(G)} \sigma(G, A, L, H, v).
\]

Assume, towards a contradiction, that a tuple $(k, G, A, L, H)$ forms a counterexample which minimizes $k$, then $|V(G)|$, and then $|A|$. For brevity, we will use the following shortcuts:

\[
\sigma(v) := \sigma(G, A, L, H, v); \quad \sigma := \sigma(G, A, L, H).
\]

Thus, (2.1) is equivalent to

\[
|A| \geq \frac{3\sigma}{2}.
\]

Note that $|V(G)|$ and $|A|$ are both positive. Indeed, if $V(G) = \emptyset$, then $J(G, A)$ is just a clique with vertex set $A$, so its DP-chromatic number is $|A|$. If, on the other hand, $A = \emptyset$, then (2.1) implies that $|L(v)| \geq \deg_G(v) + 1$ for all $v \in V(G)$, so an $(L, H)$-coloring of $G$ can be constructed greedily. Furthermore, $\chi(G) = k$, since otherwise we could have used the same $(G, A, L, H)$ with a smaller value of $k$.

**Claim 2.1.1.** For every $v \in V(G)$, the graph $J(G - v, A)$ is $(L, H)$-colorable.

**Proof.** Consider any $v_0 \in V(G)$ and let $G' := G - v_0$. For all $v \in V(G')$, $\deg_{G'}(v) \leq \deg_G(v)$, and thus $\sigma(G', A, L, H, v) \leq \sigma(v)$. Therefore,

\[
\frac{3}{2} \sigma(G', A, L, H) \leq \frac{3\sigma}{2} \leq |A|.
\]

By the minimality of $|V(G)|$, the conclusion of Theorem 2.1 holds for $(k', G', A, L, H)$, i.e., $J(G', A)$ is $(L, H)$-colorable, as claimed. \hfill \Box

**Corollary 2.1.2.** For every $v \in V(G)$,

\[
\sigma(v) = \deg_G(v) + |A| - |L(v)| + 1 > 0.
\]
**Proof.** Suppose that for some \(v_0 \in V(G)\),
\[
\deg_G(v_0) + |A| - |L(v_0)| + 1 \leq 0,
\]
i.e.,
\[
|L(v_0)| \geq \deg_G(v_0) + |A| + 1.
\]
Using **Claim 2.1.1**, fix any \((L, H)\)-coloring \(I\) of \(J(G - v_0, A)\). Since \(v_0\) still has at least
\[
|L(v_0)| - (\deg_G(v_0) + |A|) \geq 1
\]
available colors, \(I\) can be extended to an \((L, H)\)-coloring of \(J(G, A)\) greedily; a contradiction. \(\square\)

**Claim 2.1.3.** For every \(v \in V(G)\) and \(x \in \bigcup_{a \in A} L(a)\), there is \(y \in L(v)\) such that \(xy \in E(H)\).

**Proof.** Suppose that for some \(a_0 \in A, x_0 \in L(a_0)\), and \(v_0 \in V(G)\), we have \(L(v_0) \cap N_H(x_0) = \emptyset\). Let \(A' := A \setminus \{a_0\}\), and for every \(w \in V(G) \cup A'\), let \(L'(w) := L(w) \setminus N_H(x_0)\). Note that for all \(a \in A'\),
\[
|L'(a)| \geq |A'| + k, \text{ and for all } v \in V(G), \sigma(G, A', L', v) \leq \sigma(v).
\]
Moreover, by the choice of \(x_0\),
\[
|L'(v_0)| = |L(v_0)|, \text{ which due to Corollary 2.1.2, yields } \sigma(G, A', L', v_0) \leq \sigma(v_0) - 1.
\]
This implies \(\sigma(G, A', L', H) \leq |A| - 1\), and thus
\[
\frac{3}{2} \sigma(G, A', L', H) \leq \frac{3(\sigma - 1)}{2} \leq |A| - \frac{3}{2} < |A'|.
\]
By the minimality of \(|A|\), the conclusion of Theorem 2.1 holds for \((k, G, A', L', H)\), i.e., the graph \(J(G, A')\) is \((L', H)\)-colorable. By the definition of \(L'\), for any \((L', H)\)-coloring \(I\) of \(J(G, A')\), \(I \cup \{x_0\}\) is an \((L, H)\)-coloring of \(J(G, A)\). This is a contradiction. \(\square\)

**Corollary 2.1.4.** \(k \geq 2\).

**Proof.** Let \(v \in V(G)\) and consider any \(a \in A\). Since, by **Claim 2.1.3**, each \(x \in L(a)\) has a neighbor in \(L(v)\), we have
\[
|L(v)| \geq |L(a)| \geq |A| + k.
\]
Using Corollary 2.1.2, we obtain
\[
0 \leq \deg_G(v) + |A| - |L(v)| \leq \deg_G(v) - k,
\]
i.e., \(\deg_G(v) \geq k\). Since \(V(G) \not= \emptyset, k \geq 1\), which implies \(\deg_G(v) \geq 1\). But then \(\chi(G) \geq 2\), as desired. \(\square\)

**Claim 2.1.5.** \(H\) does not contain a walk of the form \(x_0 - y_0 - x_1 - y_1 - x_2\), where

- \(x_0, x_1, x_2 \in \bigcup_{a \in A} L(a)\);
- \(y_0, y_1 \in \bigcup_{v \in V(G)} L(v)\);
- \(x_0 \neq x_1 \neq x_2\) and \(y_0 \neq y_1\) (but it is possible that \(x_0 = x_2\));
- the set \(\{x_0, x_1, x_2\}\) is independent in \(H\).

**Proof.** Suppose that such a walk exists and let \(a_0, a_1, a_2 \in A\) and \(v_0, v_1 \in V(G)\) be such that \(x_0 \in L(a_0), y_0 \in L(v_0), x_1 \in L(a_1), y_1 \in L(v_1), \text{ and } x_2 \in L(a_2)\). Let \(A' := A \setminus \{a_0, a_1, a_2\}\), and for every \(w \in V(G) \cup A'\), let \(L'(w) := L(w) \setminus N_H(x_0, x_1, x_2)\). Since \(\{x_0, x_1, x_2\}\) is an independent set, for all \(a \in A'\),
\[
|L'(a)| \geq |A'| + k, \text{ while for all } v \in V(G), \sigma(G, A', L', v) \leq \sigma(v).
\]
Moreover, since for each \(i \in \{0, 1\}\), the set \(\{x_0, x_1, x_2\}\) contains two distinct neighbors of \(y_i\), we have \(\sigma(G, A', L', v_i) \leq \sigma(v_i) - 1\). Therefore,
\[
\sigma(G, A', L', H) \leq |A| - 2, \text{ and thus }
\]
\[
\frac{3}{2} \sigma(G, A', L', H) \leq \frac{3(\sigma - 2)}{2} \leq |A| - 3 \leq |A'|.
\]
By the minimality of \(|A|\), the conclusion of Theorem 2.1 holds for \((k, G, A', L', H)\), i.e., the graph \(J(G, A')\) is \((L', H)\)-colorable. By the definition of \(L'\), for any \((L', H)\)-coloring \(I\) of \(J(G, A')\), \(I \cup \{x_0, x_1, x_2\}\) is an \((L, H)\)-coloring of \(J(G, A)\). This is a contradiction. \(\square\)
Due to Corollary 2.1.4, we can choose a pair of disjoint independent sets $U_0, U_1 \subset V(G)$ such that $\chi(G - U_0) = \chi(G - U_1) = k - 1$. Choose arbitrary elements $a_1 \in A$ and $x_1 \in L(a_1)$. By Claim 2.1.3, for each $u \in U_0 \cup U_1$, there is a unique element $y(u) \in L(u)$ adjacent to $x_1$ in $H$ (the uniqueness of $y(u)$ follows from the definition of a cover). Let

$$I_0 := \{y(u) : u \in U_0\} \quad \text{and} \quad I_1 := \{y(u) : u \in U_1\}.$$ 

Since $U_0$ and $U_1$ are independent sets in $G$, $I_0$ and $I_1$ are independent sets in $H$.

**Claim 2.1.6.** There exists an element $a_0 \in A \setminus \{a_1\}$ such that $L(a_0) \cap N_H(I_0) \not\subseteq N_H(x_1)$.

**Proof.** Assume that for all $a \in A \setminus \{a_1\}$, we have $L(a) \cap N_H(I_0) \subseteq N_H(x_1)$. Let $G' := G - U_0$, and for each $w \in V(G') \cup A$, let $L'(w) := L(w) \setminus N_H(I_0)$. By the definition of $I_0$, $L'(a_1) = L(a_1) \setminus \{x_1\}$, so $|L'(a_1)| = |L(a_1)| - 1 \geq |A| + (k - 1)$.

On the other hand, by our assumption, for each $a \in A \setminus \{a_1\}$, we have

$$|L'(a)| = |L(a) \setminus N_H(I_0)| \geq |L(a) \setminus N_H(x_1)| \geq |L(a)| - 1 \geq |A| + (k - 1).$$

Since for all $v \in V(G)$, $\sigma(G', A', L', H, v) \leq \sigma(v)$, the minimality of $k$ implies the conclusion of Theorem 2.1 for $(k - 1, G', A', L, H)$; in other words, the graph $J(G', A)$ is $(L', H)$-colorable. By the definition of $L'$, for any $(L', H)$-coloring $I$ of $J(G', A)$, $I \cup I_0$ is an $(L, H)$-coloring of $J(G, A)$; this is a contradiction. \(\square\)

Using Claim 2.1.6, fix some $a_0 \in A \setminus \{a_1\}$ satisfying $L(a_0) \cap N_H(I_0) \not\subseteq N_H(x_1)$, and choose any $x_0 \in (L(a_0) \cap N_H(I_0)) \setminus N_H(x_1)$.

Since $x_0 \in N_H(I_0)$, we can also choose $y_0 \in I_0$ so that $x_0y_0 \in E(H)$.

**Claim 2.1.7.** $x_0 \not\in N_H(I_1)$.

**Proof.** If there is $y_1 \in I_1$ such that $x_0y_1 \in E(H)$, then $x_0 - y_0 - x_1 - y_1 - x_0$ is a walk in $H$ whose existence is ruled out by Claim 2.1.5. \(\square\)

**Claim 2.1.8.** There is an element $a_2 \in A \setminus \{a_0, a_1\}$ such that $L(a_2) \cap N_H(I_1) \not\subseteq N_H(x_0, x_1)$.

**Proof.** The proof is almost identical to the proof of Claim 2.1.6. Assume that for all $a \in A \setminus \{a_0, a_1\}$, we have $L(a) \cap N_H(I_1) \subseteq N_H(x_0, x_1)$. Let $G' := G - U_1$, $A' := A \setminus \{a_0\}$, and for each $w \in V(G') \cup A'$, let $L'(w) := L(w) \setminus N_H(x_0 \cup I_1)$. By the definition of $I_1$, $L(a_1) \cap N_H(I_1) = \{x_1\}$, so $|L'(a_1)| = |L(a_1)| - 2 \geq |A| + k - 2 = |A'| + (k - 1)$.

On the other hand, by our assumption, for each $a \in A \setminus \{a_0, a_1\}$, we have $|L'(a)| \geq |L(a) \setminus N_H(x_0, x_1)| \geq |L(a)| - 2 \geq |A| + k - 2 = |A'| + (k - 1)$.

Since for all $v \in V(G)$, $\sigma(G', A', L', H, v) \leq \sigma(v)$, the minimality of $k$ implies the conclusion of Theorem 2.1 for $(k - 1, G', A', L', H)$; in other words, the graph $J(G', A')$ is $(L', H)$-colorable. By the definition of $L'$, for any $(L', H)$-coloring $I$ of $J(G', A)$, $I \cup \{x_0\} \cup I_1$ is an $(L, H)$-coloring of $J(G, A)$. This is a contradiction. \(\square\)

Now we are ready to finish the proof of Theorem 2.1. Fix some $a_2 \in A \setminus \{a_0, a_1\}$ satisfying $L(a_2) \cap N_H(I_1) \not\subseteq N_H(x_0, x_1)$, and choose any $x_2 \in (L(a_2) \cap N_H(I_1)) \setminus N_H(x_0, x_1)$.

Since $x_2 \in N_H(I_1)$, there is $y_1 \in I_1$ such that $x_2y_1 \in E(H)$. Then $x_0 - y_0 - x_1 - y_1 - x_2$ is a walk in $H$ contradicting the conclusion of Claim 2.1.5. \(\blacksquare\)
Now it is easy to derive Theorem 1.5. Indeed, let \( G \) be a graph with \( n \) vertices, \( m \) edges, and chromatic number \( k \), let \( A \) be a finite set disjoint from \( V(G) \), and let \((L, H)\) be a cover of \( J(G, A) \) such that for all \( v \in V(G) \) and \( a \in A \), \(|L(v)| = |L(a)| = \chi(J(G, A)) = |A| + k \). Note that

\[
\frac{3}{2} \sum_{u \in V(G)} \max \{ \deg_G(u) - |L(u)| + |A| + 1, 0 \} = \frac{3}{2} \sum_{u \in V(G)} \max \{ \deg_G(u) - k + 1, 0 \}.
\]

If \(|A| \geq 3m\), then

\[
\frac{3}{2} \sum_{u \in V(G)} \max \{ \deg_G(u) - k + 1, 0 \} \leq \frac{3}{2} \sum_{u \in V(G)} \deg_G(v) = 3m \leq |A|,
\]

so Theorem 2.1 implies that \( J(G, A) \) is \((L, H)\)-colorable, and hence \( Z_{DP}(G) \leq 3m \). Moreover, if \( \delta(G) \geq k - 1 \), then

\[
\frac{3}{2} \sum_{u \in V(G)} \max \{ \deg_G(u) - k + 1, 0 \} \leq \frac{3}{2} \sum_{v \in V(G)} (\deg_G(v) - k + 1) = 3m - \frac{3}{2}(k - 1)n,
\]

so \( Z_{DP}(G) \leq 3m - \frac{3}{2}(k - 1)n \), as desired. Finally, Corollary 1.6 follows from Theorem 1.5 and the fact that an \( n \)-vertex graph can have at most \( \binom{n}{2} \leq n^2/2 \) edges.

3. Proof of Theorem 1.7

We will prove the following precise version of Theorem 1.7:

**Theorem 3.1.** For all even \( n \in \mathbb{N} \), \( Z_{DP}(n) \geq n^2/4 - n \).

**Proof.** Let \( n \in \mathbb{N} \) be even and let \( k := n/2 - 1 \). Note that \( n^2/4 - n = k^2 - 1 \). Thus, it is enough to exhibit an \( n \)-vertex bipartite graph \( G \) and a cover \((L, H)\) of \( J(G, k^2 - 2) \) such that \(|L(u)| = k^2 \) for all \( u \in V(J(G, k^2 - 2)) \), yet \( J(G, k^2 - 2) \) is not \((L, H)\)-colorable.

Let \( G \cong K_{n/2,n/2} \) be an \( n \)-vertex complete bipartite graph with parts \( X = \{x_0, \ldots, x_{k-1}\} \) and \( Y = \{y_0, \ldots, y_{k-1}\} \), where the indices \( 0, \ldots, k - 1 \) are viewed as elements of the additive group \( \mathbb{Z}_k \) of integers modulo \( k \). Let \( A \) be a set of size \( k^2 - 2 \) disjoint from \( X \cup Y \). For each \( u \in X \cup Y \cup A \), let \( L(u) := \{u\} \times \mathbb{Z}_k \times \mathbb{Z}_k \). Let \( H \) be the graph with vertex set \((X \cup Y \cup A) \times \mathbb{Z}_k \times \mathbb{Z}_k \) in which the following pairs of vertices are adjacent:

- \((u, i, j)\) and \((u, i', j')\) for all \( u \in X \cup Y \cup A \) and \( i, j, i', j' \in \mathbb{Z}_k \) such that \( i \neq j \).
- \((i, j, u)\) and \((i, j, u)\) for all \( i, j, u \in \mathbb{Z}_k \).
- \((x_t, i, j)\) and \((y_t, i + s, j + t)\) for all \( s, t, i, j \in \mathbb{Z}_k \).

It is easy to see that \((L, H)\) is a cover of \( J(G, A) \). We claim that \( J(G, A) \) is not \((L, H)\)-colorable. Indeed, suppose that \( I \) is an \((L, H)\)-coloring of \( J(G, A) \). For each \( u \in X \cup Y \cup A \), let \((i(u), j(u))\) be the unique elements of \( \mathbb{Z}_k \) such that \((u, i(u), j(u)) \in I \). By the construction of \( H \) and since \( I \) is an independent set, we have

\[
(i(u), j(u)) \neq (i(a), j(a))
\]

for all \( u \in X \cup Y \cup A \). Since all the \( k^2 - 2 \) pairs \((i(a), j(a))\) for \( a \in A \) are pairwise distinct, \((i(u), j(u))\) can take at most 2 distinct values as \( u \) is ranging over \( X \cup Y \). One of those 2 values is \((i(y), j(y))\), and if \( u \in X \), then

\[
(i(u), j(u)) \neq (i(y), j(y)),
\]

so the value of \((i(u), j(u))\) must be the same for all \( u \in X \); let us denote it by \((i, j)\). Similarly, the value of \((i(u), j(u))\) is the same for all \( u \in Y \), and we denote it by \((i', j')\).

It remains to notice that the vertices \((x_{i-1}, i, j)\) and \((y_{i-1}, i', j')\) are adjacent in \( H \), so \( I \) is not an independent set.

Now we can prove Corollary 1.8:
Proof of Corollary 1.8. First, suppose that $G$ is an $n$-vertex graph with $\chi(G) = r$ that maximizes the difference $\chi_{DP}(G) - \chi(G)$. Adding edges to $G$ if necessary, we may arrange $G$ to be a complete $r$-partite graph. Assuming $2r > n$, at least $2r - n$ of the parts must be of size 1, i.e., $G$ is of the form $J(G', 2r - n)$ for some $(n - r)$-vertex graph $G'$. By Corollary 1.6, we have $\chi_{DP}(G) = \chi(G)$ as long as $2r - n \geq 6(n - r)^2$, which holds for all $r \geq n - (1/\sqrt{6} - o(1))\sqrt{n}$. This establishes the upper bound $r(n) \leq n - \Omega(\sqrt{n})$.

On the other hand, due to Theorem 1.7, for each $n$, we can find a graph $G$ with $s$ vertices, where $s \leq (2 + o(1))\sqrt{n}$, such that $\chi_{DP}(J(G, n - s)) > \chi(J(G, n - s))$. Since $J(G, n - s)$ is an $n$-vertex graph, we get

$$r(n) > \chi(J(G, n - s)) = \chi(G) + n - s \geq n - (2 + o(1))\sqrt{n} = n - O(\sqrt{n}).$$

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