A stability version for a theorem of Erdős on nonhamiltonian graphs

Zoltán Füredi a, Alexandr Kostochka b,c,*, Ruth Luo b

a Alfréd Rényi Institute of Mathematics, Hungary
b University of Illinois at Urbana–Champaign, Urbana, IL 61801, United States
c Sobolev Institute of Mathematics, Novosibirsk 630090, Russia

A B S T R A C T

Let \( n, d \) be integers with \( 1 \leq d \leq \left\lfloor \frac{n-1}{2} \right\rfloor \), and set \( h(n, d) := \binom{n-d}{2} + d^2 \) and \( e(n, d) := \max\{h(n, d), h(n, \left\lfloor \frac{n-1}{2} \right\rfloor)\} \). Because \( h(n, d) \) is quadratic in \( d \), there exists a \( d_0(n) = (n/6) + O(1) \) such that

\[
e(n,1) > e(n,2) > \cdots > e(n,d_0) = e(n,d_0+1) = \cdots = e\left(n,\left\lfloor \frac{n-1}{2} \right\rfloor\right).
\]

A theorem by Erdős states that for \( d \leq \left\lfloor \frac{n-1}{2} \right\rfloor \), any \( n \)-vertex nonhamiltonian graph \( G \) with minimum degree \( \delta(G) \geq d \) has at most \( e(n,d) \) edges, and for \( d > d_0(n) \) the unique sharpness example is simply the graph \( K_n - E(K_{\left\lceil (n+1)/2 \right\rceil}) \). Erdős also presented a sharpness example \( H_{n,d} \) for each \( 1 \leq d \leq d_0(n) \).

We show that if \( d < d_0(n) \) and a 2-connected, nonhamiltonian \( n \)-vertex graph \( G \) with \( \delta(G) \geq d \) has more than \( e(n,d+1) \) edges, then \( G \) is a subgraph of \( H_{n,d} \). Note that \( e(n,d) - e(n,d+1) = n - 3d - 2 \geq n/2 \) whenever \( d < d_0(n) - 1 \).

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1. Introduction

We use standard notation. In particular, \( V(G) \) denotes the vertex set of a graph \( G \), \( E(G) \) denotes the edge set of \( G \), and \( e(G) = |E(G)| \). Also, if \( v \in V(G) \), then \( N(v) \) denotes the neighborhood of \( v \) and \( d(v) = |N(v)| \). Ore [3] proved the following Turán-type result:

**Theorem 1 (Ore [3]).** If \( G \) is a nonhamiltonian graph on \( n \) vertices, then \( e(G) \leq \binom{n-1}{2} + 1 \).

This bound is achieved only for the \( n \)-vertex graph obtained from the complete graph \( K_{n-1} \) by adding a vertex of degree 1. Erdős [2] refined the bound in terms of the minimum degree of the graph:

**Theorem 2 (Erdős [2]).** Let \( n, d \) be integers with \( 1 \leq d \leq \left\lfloor \frac{n-1}{2} \right\rfloor \), and set \( h(n, d) := \binom{n-d}{2} + d^2 \). If \( G \) is a nonhamiltonian graph on \( n \) vertices with minimum degree \( \delta(G) \geq d \), then

\[
e(G) \leq \max\left\{h(n,d), h\left(n,\left\lfloor \frac{n-1}{2} \right\rfloor\right)\right\} =: e(n,d).
\]

This bound is sharp for all \( 1 \leq d \leq \left\lfloor \frac{n-1}{2} \right\rfloor \).

*Corresponding author at: University of Illinois at Urbana–Champaign, Urbana, IL 61801, United States.

E-mail addresses: furedi.zoltan@renyi.mta.hu (Z. Füredi), kostochka@math.uiuc.edu (A. Kostochka), ruthluo2@illinois.edu (R. Luo).
Dirac’s Theorem [1] yields that Lemma 6. Let $G$ be a saturated $n$-vertex graph with $e(G) \geq h(n, \left\lfloor \frac{n-1}{2} \right\rfloor)$. Then for some $1 \leq k \leq \left\lfloor \frac{n-1}{2} \right\rfloor$, $V(G)$ contains a subset $D$ of $k$ vertices of degree at most $k$ such that $G - D$ is a complete graph.

First, we show two facts on saturated graphs with many edges.

Lemma 6. Let $G$ be a saturated $n$-vertex graph with $e(G) > h(n, \left\lfloor \frac{n-1}{2} \right\rfloor)$. Then for some $1 \leq k \leq \left\lfloor \frac{n-1}{2} \right\rfloor$, $V(G)$ contains a subset $D$ of $k$ vertices of degree at most $k$ such that $G - D$ is a complete graph.

Proof. Since $G$ is nonhamiltonian, by Theorem 4, there exists some $1 \leq k \leq \left\lfloor \frac{n-1}{2} \right\rfloor$ such that $G$ has $k$ vertices with degree at most $k$. Pick the maximum such $k$, and let $D$ be the set of the vertices with degree at most $k$. Since $e(G) > h(n, \left\lfloor \frac{n-1}{2} \right\rfloor)$, $k < \left\lfloor \frac{n-1}{2} \right\rfloor$. So, by the maximality of $k$, $|D| = k$. 

To show the sharpness of the bound, for $n, d \in \mathbb{N}$ with $d \leq \left\lfloor \frac{n-1}{2} \right\rfloor$, consider the graph $H_{n,d}$ obtained from a copy of $K_{n-d}$, say with vertex set $A$, by adding $d$ vertices of degree $d$ each of which is adjacent to the same $d$ vertices in $A$. An example of $H_{11,3}$ is given in Fig. 1.

By construction, $H_{n,d}$ has minimum degree $d$, is nonhamiltonian, and $e(H_{n,d}) = \left( \frac{n-d}{2} \right) + d^2 = h(n, d)$. Elementary calculation shows that $h(n, d) > h(n, \left\lfloor \frac{n-1}{2} \right\rfloor)$ in the range $1 \leq d \leq \left\lfloor \frac{n-1}{2} \right\rfloor$ if and only if $d < (n + 1)/6$ and $n$ is odd or $d < (n + 4)/6$ and $n$ is even. Hence there exists $d_0 := d_0(n)$ such that

$$e(n, 1) > e(n, 2) > \cdots > e(n, d_0) = e(n, d_0 + 1) = \cdots = e(n, \left\lfloor \frac{n-1}{2} \right\rfloor),$$

where $d_0(n) := \left\lfloor \frac{n+1}{4} \right\rfloor$ if $n$ is odd, and $d_0(n) := \left\lceil \frac{n+4}{6} \right\rceil$ if $n$ is even. Let $H_{n,d}'$ denote the graph that is an edge-disjoint union of two complete graphs $K_{n-d}$ and $K_{d+1}$ sharing one vertex.

The result of this note is the following refinement of Theorem 2.

Theorem 3. Let $n \geq 3$ and $d \leq \left\lfloor \frac{n-1}{2} \right\rfloor$. Suppose that $G$ is an $n$-vertex nonhamiltonian graph with minimum degree $\delta(G) \geq d$ such that

$$e(G) > e(n, d + 1) = \max \left\{ h(n, d + 1), h(n, \left\lfloor \frac{n-1}{2} \right\rfloor) \right\}. $$

(1)

(So we have $d < d_0(n)$.) Then $G$ is a subgraph of either $H_{n,d}$ or $H_{n,d}'$.

This is a stability result in the sense that for $d < n/6$, each 2-connected, nonhamiltonian $n$-vertex graph with minimum degree at least $d$ and “close” to $h(n, d)$ edges is a subgraph of the extremal graph $H_{n,d}$. Note that $h(n, d) - h(n, d + 1) = n - 3d - 2$ is at least $n/2$ for $d < d_0 - 1$. Note also that $e(H_{n,d}') > e(n, d + 1)$ only when $d = O(\sqrt{n})$.

We will use the following well-known theorems of Pósa.

Theorem 4 (Pósa [4]). Let $n \geq 3$. If $G$ is a nonhamiltonian $n$-vertex graph, then there exists $1 \leq k \leq \left\lfloor \frac{n-1}{2} \right\rfloor$ such that $G$ has a set of $k$ vertices with degree at most $k$.

Theorem 5 (Pósa [5]). Let $n \geq 3$, $1 \leq \ell < n$ and let $G$ be an $n$-vertex graph such that $d(u) + d(v) \geq n + \ell$ for every non-edge $uv$ in $G$. Then for every linear forest $F$ with $\ell$ edges contained in $G$, the graph $G$ has a hamiltonian cycle containing all edges of $F$.

2. Proof of Theorem 3

Call a graph $G$ saturated if $G$ is nonhamiltonian but for each $uv \notin E(G)$, $G + uv$ has a hamiltonian cycle. Ore’s proof [3] of Dirac’s Theorem [1] yields that

for every $n$-vertex saturated graph $G$ and for each $uv \notin E(G)$, $d(u) + d(v) \leq n - 1$. (2)

First we show two facts on saturated graphs with many edges.

Lemma 6. Let $G$ be a saturated $n$-vertex graph with $e(G) > h(n, \left\lfloor \frac{n-1}{2} \right\rfloor)$. Then for some $1 \leq k \leq \left\lfloor \frac{n-1}{2} \right\rfloor$, $V(G)$ contains a subset $D$ of $k$ vertices of degree at most $k$ such that $G - D$ is a complete graph.

Proof. Since $G$ is nonhamiltonian, by Theorem 4, there exists some $1 \leq k \leq \left\lfloor \frac{n-1}{2} \right\rfloor$ such that $G$ has $k$ vertices with degree at most $k$. Pick the maximum such $k$, and let $D$ be the set of the vertices with degree at most $k$. Since $e(G) > h(n, \left\lfloor \frac{n-1}{2} \right\rfloor)$, $k < \left\lfloor \frac{n-1}{2} \right\rfloor$. So, by the maximality of $k$, $|D| = k$. 

Fig. 1. $H_{11,3}$. 

Suppose there exist \( x, y \in V(G) - D \) such that \( xy \notin E(G) \). Among all such pairs, choose \( x \) and \( y \) with the maximum \( d(x) \). Since \( y \notin D \), \( d(y) > k \). Let \( D' := V(G) - N(x) - \{x\} \) and \( k' := |D'| = n - 1 - d(x) \). By (2),

\[
d(z) \leq n - 1 - d(x) = k' \quad \text{for all } z \in D'.
\]

So \( D' \) is a set of \( k' \) vertices of degree at most \( k' \). Since \( y \in D' \), \( k' \geq d(y) > k \). Thus by the maximality of \( k \), we get \( k' = n - 1 - d(x) = \lceil \frac{n-1}{2} \rceil \). Equivalently, \( d(x) < \lceil \frac{n-1}{2} \rceil \). For all \( z \in D' + \{x\} \), either \( z \in D \) where \( d(z) \leq k \leq \lceil \frac{n-1}{2} \rceil \), or \( z \in V(G) - D \), and so \( d(z) \leq d(x) \leq \lceil \frac{n-1}{2} \rceil \). It follows that \( e(G) \leq h(n, \lfloor \frac{n-1}{2} \rfloor) \), a contradiction. \( \square \)

**Lemma 7.** Under the conditions of Lemma 6, if \( k = \delta(G) \), then \( G = H_{n,\delta(G)} \) or \( G = H'_{n,\delta(G)} \).

**Proof.** Set \( d := \delta(G) \), and let \( D \) be a set of \( d \) vertices with degree at most \( d \). Let \( u \in D \). Since \( \delta(G) \geq |D| = d \), \( u \) has a neighbor \( w \in V(G) - D \). Consider any \( v \in D - \{u\} \). By Lemma 6, \( w \) is adjacent to all of \( V(G) - D - \{w\} \). It also is adjacent to \( u \), therefore its degree is at least \( n - d \). We obtain

\[
d(w) + d(v) \geq (n - d) + d = n.
\]

Then by (2), \( w \) is adjacent to \( v \), and hence \( w \) is adjacent to all vertices of \( D \).

Let \( W \) be the set of vertices in \( V(G) - D \) having a neighbor in \( D \). We have obtained that \( W \neq \emptyset \) and

\[
N(u) \cap (V(G) - D) = W \quad \text{for all } u \in D.
\]

Let \( G' := G[D \cup W] \). If \( |W| = 1 \), then \( G = H'_{n,d} \). If \( |V(G')| = 2d \), then by (4), each vertex \( u \in D \) has the same \( d \) neighbors in \( V(G) - D \). Because \( d(u) = d, D \) is an independent set. Thus \( G = H_{n,d} \). Otherwise, \( d + 2 \leq |V(G')| \leq 2d - 1, |D| \geq 2. \)

Fix a pair of vertices \( w_1, w_2 \in W \). For any \( x, y \in V(G') \),

\[
d(x) + d(y) \geq d + d \geq |V(G')| + 1.
\]

Therefore by Theorem 5, \( G' \) has a hamiltonian cycle \( C \) that uses the edge \( w_1w_2 \). Since \( G'' := G - (V(G') - \{w_1, w_2\}) \) is a complete graph, it contains a hamiltonian \( w_1, w_2 \)-path \( P \). Then \( P \cup (C - w_1w_2) \) is a hamiltonian cycle of \( G \), a contradiction. \( \square \)

**Proof of Theorem 3.** Suppose that an \( n \)-vertex, nonhamiltonian graph \( G \) satisfies the constraints of Theorem 3 for some \( 1 \leq d \leq \lfloor \frac{n-1}{2} \rfloor \). We may assume \( G \) is saturated, since if a graph containing \( G \) is a subgraph of \( H_{n,d} \) or \( H'_{n,d} \), then \( G \) is as well.

By Lemma 6, \( G \) has a set \( D \) of \( k \leq \lfloor \frac{n-1}{2} \rfloor \) vertices with degree at most \( k \) such that \( G - D \) is a complete graph. Therefore \( e(G) \leq (\frac{n-k}{2}) + k^2 = h(n, k) \). If \( k \geq d + 1 \), then \( e(G) \leq \max\{h(n, d + 1), h(n, \lfloor \frac{n-1}{2} \rfloor)\} = e(n, d + 1) \), a contradiction. Thus \( k \leq d \). Furthermore, \( k \geq \delta(G) \geq d \), and hence \( k = d \). Also, since \( e(G) > h(n, \lfloor \frac{n-1}{2} \rfloor) \), we have \( d + 1 \leq d_0(n) \leq (n + 8)/6 \). Applying Lemma 7 completes the proof. \( \square \)

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