Adding Edges to Increase the Chromatic Number of a Graph

ALEXANDR KOSTOCHKA† and JAROSLAV NEŠETŘIL‡

1Department of Mathematics, University of Illinois, Urbana, IL 61801, USA
and
Sobolev Institute of Mathematics, Novosibirsk, Russia
(e-mail: kostochk@math.uiuc.edu)

2Computer Science Institute of Charles University, Faculty of Mathematics and Physics,
Charles University, Malostranské nám. 25, Praha 1, Czech Republic
(e-mail nesetril@iuuk.mff.cuni.cz)

Received 18 April 2015; revised 4 February 2016; first published online 31 March 2016

If \( n \geq k + 1 \) and \( G \) is a connected \( n \)-vertex graph, then one can add \( \binom{k}{2} \) edges to \( G \) so that the resulting graph contains the complete graph \( K_{k+1} \). This yields that for any connected graph \( G \) with at least \( k + 1 \) vertices, one can add \( \binom{k}{2} \) edges to \( G \) so that the resulting graph has chromatic number \( > k \). A long time ago, Bollobás suggested that for every \( k \geq 3 \) there exists a \( k \)-chromatic graph \( G_k \) such that after adding to it any \( \binom{k}{2} - 1 \) edges, the chromatic number of the resulting graph is still \( k \). In this note we prove this conjecture.

2010 Mathematics subject classification: Primary 05C15
Secondary 05C35

1. Introduction

This note is another contribution to the old theme of sparse graphs with high chromatic number [4].

For a positive integer \( k \) and a connected graph \( G \) with at least \( k + 1 \) vertices, let \( f(G,k) \) denote the minimum number \( m \) of edges such that after adding \( m \) edges (and any number of vertices) to \( G \) we obtain a graph with chromatic number at least \( k + 1 \). Since \( G \) is connected, we can add \( \binom{k}{2} \) edges to some subtree of \( G \) with \( k \) edges so that the resulting graph contains the complete graph \( K_{k+1} \). Thus \( f(G,k) \leq \binom{k}{2} \) for every connected graph \( G \) with at least \( k + 1 \) vertices. One may expect that if in addition \( G \) is \( k \)-chromatic,

† Supported in part by NSF grant DMS-1266016 and by grants 15-01-05867 and 16-01-00499 of the Russian Foundation for Basic Research.
‡ Supported in part by the Project LL1201 ERC-CZ CORES and by CE-ITI P202/12/G061 of GAČR.
then \( f(G, k) < \binom{k}{2} \). However, in the 1970s, Bollobás [1] suggested that for every \( k \geq 3 \) there exists a \( k \)-chromatic connected graph \( G_k \) with at least \( k + 1 \) vertices such that \( f(G_k, k) = \binom{k}{2} \). The goal of this note is to prove this conjecture.

In [3], the authors used uniform hypergraphs of large girth to modify the classic construction of Tutte (see [2]) to obtain a sequence of sparse graphs \( G(k, g) \) with chromatic number \( k \) and girth \( g \). We shall use these graphs to prove our result. The properties of these graphs we shall need are collected in the following lemma.

**Lemma 1.1.** Let \( k \geq 4 \) and \( g \geq 3 \). Then \( G = G(k, g) \) has chromatic number \( k \) and girth \( g \); furthermore, its vertex set has partition \( V_3 \cup V_4 \cup \cdots \cup V_k \) such that

(a) the sets \( V_4, \ldots, V_k \) are independent, and \( G[V_3] \) is the disjoint union of cycles;

(b) for all \( 3 \leq i < j \leq k \), each vertex \( v \in V_i \) has exactly one neighbour in \( V_j \); in particular, if \( v \in V_3 \), then \( d(v) = k - 1 \).

Our main result, stated next, will be proved in Section 2.

**Theorem 1.2.** Let \( k \geq 4, g \geq k^4, G = (V, E) = G(k, g) \) and let \( H = (V', E') \) be a graph with \( \chi(H) \geq k + 1 \). Then \( |E(H) \setminus E(G(k, g))| \geq \binom{k}{2} \).

### 2. Proof of Theorem 1.2

As every graph of chromatic number \( k + 1 \) contains a subgraph of minimum degree \( k \), it suffices to show that if \( H \) has minimum degree \( k \) then

\[
|E(H) \setminus E(G(k, g))| \geq \binom{k}{2}.
\]  

Note that \( |V'| \geq \delta(H) + 1 \geq k + 1 \). Assuming, as we may, that \( V' \subseteq V \), set \( G' = G[V'] \). As remarked at the beginning, (2.1) holds if \( G' \) is acyclic. Assume that \( G' \) contains a cycle. Since every cycle of \( G \) (and so of \( G' \)) contains at least \( g \geq k^4 \) vertices, \( |V'| \geq k^4 \).

For \( i = 3, \ldots, k \), set \( V'_i = V_i \cap V' \) and let \( W_i \) denote the set of vertices in \( V'_i \) incident with the edges in \( E(H') - E(G) \). In particular, \( V'_3 = W_3 \), since in \( G \) the vertices in \( V_3 \) have degree \( k - 1 \). We will prove that

\[
|W_i| \geq k^2 \text{ for some } i \text{ with } 3 \leq i \leq k.
\]

This implies \( |E'| \geq k^2/2 \), and so the theorem.

Suppose (2.2) is false, i.e., \( |W_i| < k^2 \) for every \( i \); thus our task is to arrive at a contradiction. Since

\[
|V'| = \left| \bigcup_{i=3}^{k} V'_i \right| \geq k^4,
\]

there is an \( i \) with \( |V'_i| > k^3 \) and so \( |V'_i \setminus W_i| > k^3 - k^2 \). Choose the minimum such \( i \). Then \( i > 3 \) because otherwise we would have \( k^2 > |W_3| = |V'_3| > k^3 \).

Since \( |V'_i \setminus W_i| > k^3 - k^2 \), and each vertex in \( V'_i \setminus W_i \) has at least \( k - (k - i) = i \) neighbours in \( V'_3 \cup \cdots \cup V'_{i-1} \), and no vertex in \( V'_3 \cup \cdots \cup V'_{i-1} \) has more than one neighbour
in \( V_i \supset V'_i \setminus W_i \), the set \( V'_i \setminus W_i \) has at least \( i|V'_i \setminus W_i| \) neighbours in \( V'_3 \cup \cdots \cup V'_{i-1} \). In particular,

\[
|V'_3 \cup \cdots \cup V'_{i-1}| \geq i|V'_i \setminus W_i| > i(k^3 - k^2),
\]

and so there is a \( j \in \{3, \ldots, i-1\} \) such that

\[
|V'_j| \geq \frac{ik^2(k - 1)}{i - 3} > k^3,
\]

contradicting the minimality of \( i \), and completing the proof of (2.2) and the theorem. \( \square \)

Note that the proof yields more than is claimed in the theorem. We did not make use of \( \chi(H) \geq k \); we used only that \( H \) has a subgraph of minimum degree \( k \). Briefly, if \( H \supset G(k, g) \) has minimum degree at least \( k \) then \( |E(H) \setminus E(G(k, g))| \geq (\binom{k}{2}) \), provided \( k \geq 4 \) and \( g \geq k^4 \).

Acknowledgement

We thank Béla Bollobás for fruitful discussions and a referee for suggestions greatly improving the presentation of this note.

References