COLORING SOME FINITE SETS IN $\mathbb{R}^n$

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This note relates to bounds on the chromatic number $\chi(\mathbb{R}^n)$ of the Euclidean space, which is the minimum number of colors needed to color all the points in $\mathbb{R}^n$ so that any two points at the distance 1 receive different colors. In [6] a sequence of graphs $G_n$ in $\mathbb{R}^n$ was introduced showing that
$$
\chi(\mathbb{R}^n) \geq \chi(G_n) \geq (1 + o(1)) \frac{n^2}{\pi}. 
$$
For many years, this bound has been remaining the best known bound for the chromatic numbers of some low-dimensional spaces. Here we prove that $\chi(G_n) \sim \frac{n^2}{\pi}$ and find an exact formula for the chromatic number in the case of $n = 2^k$ and $n = 2^k - 1$.

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1. Introduction

In this note, we study the classical chromatic number $\chi(\mathbb{R}^n)$ of the Euclidean space. The quantity $\chi(\mathbb{R}^n)$ is the minimum number of colors needed to color all the points in $\mathbb{R}^n$ so that any two points at a given distance $a$ receive different colors. By a well-known compactness result of Erdős and de Bruijn (see [1]), the value of $\chi(\mathbb{R}^n)$ is equal to the chromatic number of a finite distance graph $G = (V, E)$, where $V \subset \mathbb{R}^n$ and $E = \{\{x, y\} : |x - y| = a\}$.

Now we know that
$$
(1.239 \ldots + o(1))^n \leq \chi(\mathbb{R}^n) \leq (3 + o(1))^n,
$$
where the lower bound is due to the third author of this paper (see [8]) and the upper bound is due to Larman and Rogers (see [6]). Also, in [3] one can find an up-to-date table of estimates obtained for the dimensions $n \leq 12$.

It is worth noting that the linear bound $\chi(\mathbb{R}^n) \geq n + 2$ is quite simple, and the first superlinear bound was discovered by Larman, Rogers, Erdős, and Sós in [6]. They considered a family of graphs $G_n = (V_n, E_n)$ with
$$
V_n = \{x = (x_1, \ldots, x_n) : x_i \in \{0, 1\}, x_1 + \cdots + x_n = 3\},
$$
$$
E_n = \{\{x, y\} : |x - y| = 2\}.
$$
In other words, the vertices of $G_n$ are all the 3-subsets of the set $[n] = \{1, \ldots, n\}$ and two vertices $A, B$ are connected with an edge iff $|A \cap B| = 1$. Larman and Rogers [6] used an earlier result by Zs. Nagy who proved the following theorem.

**Theorem 1** [6]. Let $s$ and $t \leq 3$ be nonnegative integers and let $n = 4s + t$. Then
$$
\alpha(G_n) = \begin{cases} 
n, & \text{if } t = 0, \\
n - 1, & \text{if } t = 1, \\
n - 2, & \text{if } t = 2 \text{ or } t = 3. 
\end{cases}
$$
The standard inequality \( \chi(G_n) \geq \frac{|V_n|}{\alpha(G_n)} \) combined with the above theorem gives an obvious corollary.

**Corollary 2** [6]. Let \( s \) and \( t \leq 3 \) be nonnegative integers and let \( n = 4s + t \). Then

\[
\chi(G_n) \geq \begin{cases} 
\frac{(n-1)(n-2)}{6}, & \text{if } t = 0, \\
\frac{n(n-3)}{6}, & \text{if } t = 1, \\
\frac{n(n-1)}{6}, & \text{if } t = 2 \text{ or } t = 3.
\end{cases}
\]

The bounds from the corollary are applied to estimate from below the chromatic numbers \( \chi(R_n^{27}) \), since the vertices of \( G_n \) lie in the hyperplane \( x_1 + \cdots + x_n = 3 \). Now all these bounds are surpassed due to the consideration of some other distance graphs (see [3]). However, it could happen that actually \( \chi(G_n) \) is much bigger than the ratio \( \frac{|V_n|}{\alpha(G_n)} \). It turns out that this is not the case, and the main result of this note is as follows.

**Theorem 3.** If \( n = 2^k \) for some integer \( k \geq 2 \), then

\[
\chi(G_n) = \frac{(n-1)(n-2)}{6}.
\]

Additionally, if \( n = 2^k - 1 \) for some integer \( k \geq 2 \), then

\[
\chi(G_n) = \frac{n(n-1)}{6}.
\]

Finally, there is a constant \( c \) such that for every \( n \),

\[
\chi(G_n) \leq \frac{(n-1)(n-2)}{6} + cn.
\]

Our proof yields that \( c \leq 5.5 \). With some more work we could prove that \( c \leq 4.5 \). On the other hand, since \( n(n-1)/6 - (n-1)(n-2)/6 = (n-1)/3 \), we have \( c \geq 1/3 \).

In the next section, we prove Theorem 3.

2. Proof of Theorem 3

Easily, \( \chi(G_3) = 1 \), \( \chi(G_4) = 1 \), \( \chi(G_5) = 3 \).

Let \( f(n) := \frac{(n-1)(n-2)}{6} \). We show by induction on \( k \) that \( \chi(G_{2^k}) = f(2^k) \). For \( k = 2 \) it is trivial. Assume that for some \( k \) we established the induction hypothesis. Partition the set \([n] = [2^{k+1}]\) into the equal parts \( A_1 = \left[ \frac{n}{2} \right], A_2 = \left[ n \setminus \frac{n}{2} \right] \) of size \( 2^k \). Denote by \( U_1 \) and \( U_2 \) the sets of vertices of \( G = G_{2^{k+1}} \) lying in the sets
$A_1$ and $A_2$ respectively. By the induction assumption, each of the induced subgraphs $G[U_1]$ and $G[U_2]$ can be properly colored with at most $f(2^k)$ colors. Cover all pairs of the $2k$ elements of $A_1$ with disjoint perfect matchings $N_1, \ldots, N_{2k-1}$ and all pairs of the $2^k$ elements of $A_2$ with matchings $M_1, \ldots, M_{2^k-1}$. We form a color class $C(i, j)$ for $1 \leq i \leq 2^k - 1, 1 \leq j \leq 2^{k-1}$ as follows. Consider the matchings $N_i, M_i$ and assume that the edges are $\{u_1, u_2\}, \{u_3, u_4\}, \ldots$ in $N_i$ and $\{v_1, v_2\}, \{v_3, v_4\}, \ldots$ in $M_i$. For $j = 1, \ldots, 2^{k-1}$ let $D(i, j)$ denote the following set of 4-tuples (indices are considered modulo $2^k$):

$$\{u_1, u_2, v_{2j-1}, v_{2j}\}, \{u_3, u_4, v_{2j+1}, v_{2j+2}\}, \ldots, \{u_{2^k-1}, u_{2k}, v_{2j-3}, v_{2j-2}\}.$$

For $i = 1, \ldots, 2^k - 1$ and $j = 1, \ldots, 2^{k-1}$, the color class $C(i, j)$ is formed by the collection of triples contained in the members of $D(i, j)$. The intersection sizes are all 0 or 2, so the triples in $C(i, j)$ form an independent set in $G$. Moreover, each triple is contained in a member of some $D(i, j)$. The total number of used colors is

$$2^{k-1}(2^k - 1) + f(2^k) = 2^{2k-1} - 2^{k-1} + \frac{2^{k-1}(2^k-2)}{6} = f(2^{k+1}).$$

This proves the first statement of the theorem. Since $\chi(G_n) \leq \chi(G_{n+1})$, this together with Corollary 2 also implies the statement of the theorem for $n = 2^k - 1$.

It remains to show that there exists a constant $c$ such that $\chi(G_n) \leq \frac{n^2}{4} + cn$ for every $n$. Consider our coloring in steps.

**Step 1**: Let $n = 4s_1 + t_1$ where $t_1 \leq 3$. First, color all triples containing the elements $4s_1 + 1, \ldots, 4s_1 + t_1$ with at most $t_1(n - 1) < 3n$ colors. Now consider the set $\{4s_1\}$ and all the triples in this set. Partition $\{4s_1\}$ into $A_1 = [2s_1]$ and $A_2 = [4s_1] - [2s_1]$ and color the triples intersecting both $A_1$ and $A_2$ with $s_1(2s_1 - 1) < \frac{n}{4} \left(\frac{n}{2} - 1\right)$ colors as above.

**Step 2**: Since the triples contained in $A_1$ are disjoint from the triples contained in $A_2$, we will use for coloring the triples contained in $A_2$ the same colors and the same procedure as for the triples contained in $A_1$. Consider $A_1$. Let $n_1 = |A_1| = 2s_1 = 4s_2 + t_2$ where $t_2 \leq 3$. Since $2s_1$ is even, $t_2 \leq 2$. By construction, $n_1 \leq \frac{n}{4}$. Similarly to Step 1, color all triples containing the elements $4s_2 + 1, \ldots, 4s_2 + t_2$ with at most $t_2(n_1 - 1) < 2n_1$ colors. Partition $\{4s_2\}$ into $A_{1,1} = [2s_2]$ and $A_{1,2} = [4s_2] - [2s_2]$ and color the triples intersecting both $A_{1,1}$ and $A_{1,2}$ with at most $\frac{n}{4} \left(\frac{n}{4} - 1\right)$ new colors.

**Step i (for $i \geq 3$)**: If $2s_{i-1} \leq 2$, then Stop. Otherwise, repeat Step 2 with $[2s_{i-1}]$ in place of $[2s_1]$.

Altogether, we use at most $(3n + \frac{n(n-1)}{4}) + (\frac{2n}{2} + \frac{n(n-1)}{8}) + (\frac{2n}{4} + \frac{n(n-1)}{16}) + \cdots + 5n + \frac{n^2}{8} \cdot \frac{4}{3} = \frac{n^2}{4} + 5n = \frac{(n-1)(n-2)}{6} + 5.5n - 1/3$ colors. The theorem is proved.
3. Discussion

As we have already said, the constant 5 in the bound \( \chi(G_n) \leq \frac{n^2}{5} + 5n \) is not the best possible and can be improved. However, to find the exact value of the chromatic number is still interesting. For example, we know that \( \chi(\mathbb{R}^2) \geq 27 \) (see [3]). At the same time, \( \chi(G_{13}) \geq \left\lceil \frac{13}{12} \right\rceil = 24 \) (due to Corollary 2), and the proof of Theorem 3 applied for \( n = 13 \) yields a bound \( \chi(G_{13}) \leq 31 \).

It would be quite interesting to study more general graphs. Let \( G(n,r,s) \) be the graph whose set of vertices consists of all the \( r \)-subsets of the set \([n]\) and whose set of edges is formed by all possible pairs of vertices \( A, B \) with \(|A \cap B| = s\). Larman proved in [5] that

\[
\chi(\mathbb{R}^n) \geq \chi(G(n,5,2)) \geq \frac{n^3}{\alpha(G(n,5,2))} \geq \frac{\binom{n}{5}}{1485n^2} \sim \frac{n^3}{178200}.
\]

Thus, the main result of Larman was in finding the bound \( \alpha(G(n,5,2)) \leq (1 + o(1))1485n^2 \). However, the so-called linear algebra method ([2], see also [8]) can be directly applied here to show that \( \alpha(G(n,5,2)) \leq (1 + o(1))\left(\frac{n}{5}\right)^2 \). This substantially improves Larman’s estimate and gives \( \chi(G(n,5,2)) \geq (1 + o(1))\frac{n^3}{50} \). We do not know any further improvements on this result. On the other hand, observe that for any 3-set \( A \), the collection of 5-sets containing \( A \) forms an independent set in \( G(n,5,2) \), yielding \( \chi(G(n,5,2)) \leq \binom{n}{5} \sim \frac{n^5}{6} \). It is plausible that \( \chi(G(n,5,2)) \sim cn^3 \) with a constant \( c \in [1/60,1/6] \), but this constant is not yet found and even no better bounds for \( c \) have been published.

Furthermore, the graphs \( G(n,5,3) \) have been studied, since the best known lower bound \( \chi(\mathbb{R}^3) \geq 21 \) is due to the fact that \( \chi(G(10,5,3)) = 21 \) (see [4]). No related results concerning the case of \( n \to \infty \) have apparently been published.

Although for combinatorial geometry small values of \( n \) are of greater interest, we see that the consideration of graphs \( G(n,r,s) \) with small \( r, s \) and growing \( n \) is of its intrinsic interest, too. So assume that \( r, s \) are fixed and \( n \to \infty \). We have

\[
\chi(G(n,r,s)) \leq \min\{O(n^{r-s}),O(n^{s+1})\}.
\]

The first bound follows from Brooks’ theorem, since the maximum degree of \( G(n,r,s) \) is

\[
\binom{r}{s}\frac{n-r}{r-s} = (1+o(1))\frac{r!}{s!(r-s)!}n^{r-s}.
\]

The second bound is a simple generalization of the above-mentioned bound \( \chi(G(n,5,2)) \leq (1+o(1))n^3/6 \).

Note that the second bound can be somewhat improved. Assume \( s < r/2 \), so \( q := \lfloor (r-1)/s \rfloor \) is at least 2. Assuming that \( q \) divides \( n \), partition \([n]\) into \( q \)
equal classes, $A_1, \ldots, A_q$. Let $C$ be the family of $(s+1)$-sets that are subsets of some $A_i$. For each $B \in C$, the $r$-sets containing $B$ form an independent set in $G(n, r, s)$, and by the pigeonhole principle every $r$-set contains such $B$, hence

$$\chi(G(n, r, s)) \leq |C| = q \left( \frac{n/q}{s+1} \right) = (1 + o(1)) \frac{n^{s+1}}{q^s(s+1)!}.$$  

In particular, $\chi(G(n, 5, 2)) \leq (1 + o(1)) \frac{n^5}{24}$, which improves the previous bound $\frac{n^5}{6}$.  

It is worthwhile to look at the construction in Section 2 from a different point of view. For $n = 2^k$ we constructed a 4-uniform hypergraph $\mathcal{H}$ with the property that every 3-subset of vertices is covered exactly once. Note that $e(\mathcal{H}) = \binom{n}{3}/4$. Then we decomposed $E(\mathcal{H})$ into $\binom{n}{3}$ perfect matchings. Each matching gives a color class of our coloring. Note that instead of providing the explicit decomposition, we could have used a classical theorem of Pippenger and Spencer [7], which claims the existence of $(1 + o(1))\binom{n}{3}$ covering matchings. 

This motivates the following possible approach to the case $r = 2s + 1$. The discussion here is not a proof, it is just a sketch of a possible way to generalize our argument. Assume that we managed to construct an $(r+s)$-uniform hypergraph $\mathcal{H}$ that covers every $r$-set exactly once. Then $e(\mathcal{H}) = \binom{n}{r}/\binom{r+s}{r}$. Assume that $\mathcal{H}$ can be decomposed into $t$ hypergraphs, $\mathcal{N}_1, \ldots, \mathcal{N}_t$, such that for every $i$ and every $A, B \in \mathcal{N}_i$ we have $|A \cap B| \leq s - 1$. Then the $r$-sets covered by sets in $\mathcal{N}_i$ form an independent set, yielding $\chi(G(n, r, s)) \leq t$. Probably a generalization of the theorem of Pippenger and Spencer [7] would give $t \leq (1 + o(1))\binom{n}{r}/\binom{r+s}{r} = (1 + o(1))(s!/r!)n^{r-s}$. This bound, if true, would be asymptotically best possible, since the already mentioned linear algebra method (see [2, 8]) ensures that $\alpha(G(n, 2s+1, s)) \leq (1 + o(1))\binom{n}{s}$ and so $\chi(G(n, 2s+1, s)) \geq (1 + o(1))\binom{n}{s}/\binom{r+s}{r}$. Provided $s + 1$ is a prime power. In particular, we would get $\chi(G(n, 5, 2)) \sim \frac{n^3}{60}$.

The case of simultaneously growing $n, r, s$ has also been studied. Namely, $r \sim r'n$ and $s \sim s'n$ with any $r' \in (0, 1)$ and $s' \in (0, r')$ have been considered. This is due to the fact that the first exponential estimate to the quantity $\chi(\mathbb{R}^n)$, $\chi(\mathbb{R}^n) \geq (1.207 \cdots + o(1))^n$, was obtained by Frankl and Wilson in [2] with the help of some graphs $G(n, r, s)$ having $r \sim r'n$ and $s \sim \frac{r'}{r}n$. Lower bounds are usually based on the linear algebra (see [8]) and upper bounds can be found in [9].

References


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