On 1-improper 2-coloring of sparse graphs

O.V. Borodin\textsuperscript{a,b}, A. Kostochka\textsuperscript{c,a}, M. Yancey\textsuperscript{d,∗}

\textsuperscript{a} Sobolev Institute of Mathematics, Novosibirsk 630090, Russia
\textsuperscript{b} Novosibirsk State University, Novosibirsk 630090, Russia
\textsuperscript{c} University of Illinois at Urbana-Champaign, Urbana, IL 61801, USA
\textsuperscript{d} Department of Mathematics, University of Illinois, Urbana, IL 61801, USA

\begin{abstract}
A graph $G$ is $(1, 1)$-colorable if its vertices can be partitioned into subsets $V_1$ and $V_2$ such that every vertex in $G[V_i]$ has degree at most 1 for each $i \in \{1, 2\}$. We prove that every graph with maximum average degree at most $12/7$ is $(1, 1)$-colorable. In particular, it follows that every planar graph with girth at least 7 is $(1, 1)$-colorable. On the other hand, we construct graphs with maximum average degree arbitrarily close to $12/7$ (from above) that are not $(1, 1)$-colorable.

In fact, we establish the best possible sufficient condition for the $(1, 1)$-colorability of a graph $G$ in terms of the minimum, $\rho(G)$, of $\rho(G[S]) = |S| - 5|E(G[S])|$ over all subsets $S$ of $V(G)$. Namely, every graph $G$ with $\rho(G) ≥ 0$ is $(1, 1)$-colorable. On the other hand, we construct infinitely many non-$(1, 1)$-colorable graphs $G$ with $\rho(G) = −1$. This solves a related conjecture of Kurek and Ruciński from 1994.

© 2013 Elsevier B.V. All rights reserved.
\end{abstract}

1. Introduction

A graph $G$ is called improperly $(d_1, \ldots, d_k)$-colorable, or just $(d_1, \ldots, d_k)$-colorable, if the vertex set of $G$ can be partitioned into subsets $V_1, \ldots, V_k$ such that the graph $G[V_i]$ induced by the vertices of $V_i$ has maximum degree at most $d_i$ for all $1 ≤ i ≤ k$. This notion generalizes those of proper $k$-coloring (when $d_1 = \cdots = d_k = 0$) and $d$-improper $k$-coloring (when $d_1 = \cdots = d_k = d ≥ 1$).

The first result on $d$-improper colorings with $d > 0$ belongs to Gerencsér [13], who proved that every graph $G$ with maximum degree $\Delta(G)$ is 1-improperly $(\lceil \frac{\Delta(G)}{2} \rceil + 1)$-colorable; in particular, every subcubic graph is $(1, 1)$-colorable. This was extended by Lovász [17] as follows: every graph $G$ is $(d_1, \ldots, d_k)$-colorable whenever $(d_1+1)+\cdots+(d_k+1) ≥ \Delta(G)+1$.

These bounds are attained by the complete graphs.

As shown by Appel and Haken [12], every planar graph is 4-colorable, i.e. $(0, 0, 0, 0)$-colorable. Cowen, Cowen, and Woodall [11] proved that every planar graph is 2-improperly 3-colorable, i.e. $(2, 2, 2)$-colorable.

Another important extension of proper coloring was introduced by Vizing [19] and, independently, by Erdős, Rubin, and Taylor [12]. Suppose that for each list $L(v)$ of colors admissible for $v$ such that $|L(v)| ≥ k$, there is a proper coloring in which a color of vertex $v$ is taken from $L(v)$; then $G$ is $k$-choosable. Clearly, if $L(v)$ is the same set of cardinality $k$ for all vertices, then we have the case of proper $k$-coloring.

Borodin and Kostochka [8] extended the notion of $(d_1, \ldots, d_k)$-colorability as follows: Let $f_i$, $1 ≤ i ≤ s$, be functions from $V(G)$ to the non-negative integers. A graph $G$ is called $(f_1, \ldots, f_s)$-choosable if $V(G)$ can be partitioned into subsets $V_1, \ldots, V_s$ such that each vertex $v \in V_i$ (i.e., colored with $i$), where $1 ≤ i ≤ s$, has strictly fewer than $f_i(v)$ neighbors in $V_i$. Clearly, if

\[ f_i(v) ≤ 1 \]

for all $v \in V_i$ and $i$, then $G$ is $(1, \ldots, 1)$-colorable. This is the case for planar graphs, and $\Delta(G) + 1$ is an upper bound for $f_i(v)$ in any proper coloring of $G$.

For $s = 1$, the notion of $(1, \ldots, 1)$-choosability gives the same notion as $(1, 1)$-colorability. Another special case is $s = 2$, and the notion of $(f_1, f_2)$-choosability gives the same notion as $(d_1, d_2)$-colorability. A graph $G$ is $(d_1, d_2)$-colorable if $V(G)$ can be partitioned into subsets $V_1, V_2$ such that $\Delta(V_1) ≤ d_1$ and $\Delta(V_2) ≤ d_2$.

As shown by Vizing [19] and, independently, by Erdős, Rubin, and Taylor [12], a planar graph is $4$-colorable. For general graphs, the best known bound is $\Delta(G) + 1$.

It is not known whether there is an upper bound for the number of colors that is independent of the maximum degree.

\begin{acknowledgements}
\end{acknowledgements}

\begin{references}
\end{references}
By Borodin and Ivanova [3]: they proved that every graph $G$ (planar graph $G$) with girth $\geq 5$ is $k$-improperly 2-choosable, i.e., $(k, k)$-choosable.

A natural measure of sparseness for a graph $G$ is $mad(G) = \max \left\{ \frac{2|E(G)|}{|V(G)|-1} : H \subseteq G \right\}$, the maximum over the average degrees of the subgraphs of $G$. For planar graphs $G$ the sparseness can be measured in terms of the girth, $g(G)$, which is the length of a shortest cycle in $G$. It is an easy consequence of Euler's formula that each planar graph $G$ satisfies $mad(G) \leq \frac{2g(G)}{g(G)-2}$.

We now survey the known results on probably the simplest version of improper coloring, namely improper colorings of sparse graph with two colors, and, more generally, $k$-improper 2-choosability. Mihók [18] constructed a planar graph that is not $(k, k)$-colorable for arbitrarily large $k$. Havet and Sereni [15] proved, for every $k \geq 0$, that every graph $G$ with $mad(G) < \frac{4k+4}{k+2}$ is $k$-improperly 2-choosable, i.e, $(k, k)$-choosable.

For non-negative integers $j$ and $k$, let $F(j, k)$ denote the supremum of $x$ such that every graph $G$ with $mad(G) \leq x$ is $(j, k)$-colorable. It is easy to see that $F(0, 0) = 2$. Indeed, since the odd cycle $C_{2n-1}$ has $mad(G) = 2$ and is not $(0, 0)$-colorable, $F(0, 0) \leq 2$. On the other hand, each graph with $mad(G) < 2$ has no cycles and therefore is bipartite, i.e., $(0, 0)$-colorable.

Glebov and Zambaraeva [14] proved that every planar graph $G$ with $g(G) \geq 16$ is $(0, 1)$-colorable. This was strengthened by Borodin and Ivanova [3]: they proved that every graph $G$ with $mad(G) < \frac{7}{3}$ is $(0, 1)$-colorable, which implies that every planar graph $G$ with $g(G) \geq 14$ is $(0, 1)$-colorable. Borodin and Kostochka [9] proved that $F(0, 1) = \frac{12}{5}$. In particular, this implies that every planar graph $G$ with $g(G) \geq 12$ is $(0, 1)$-colorable.

For each integer $k \geq 2$, Borodin et al. [5] proved that every graph $G$ with $mad(G) < \frac{4k+4}{k+2} = 3 - \frac{2}{k+2}$ is $(0, k)$-colorable. On the other hand, for all $k \geq 2$ [5] presents non-$(0, k)$-colorable graphs with $mad(G)$ arbitrarily close to $\frac{4k+4}{k+2} = 3 - \frac{1}{k+1}$.

Recently, it was proved by Borodin et al. [6] that every graph $G$ with $mad(G) < \frac{10k+22}{3k+9}$, where $k \geq 2$, is $(1, k)$-colorable. On the other hand, [6] presents a construction of non-$(1, 1)$-colorable graphs whose maximum average degree is arbitrarily close to $\frac{14k+31}{6k+11}$.

Borodin and Kostochka [10] obtained an exact result for a wide range of $j$ and $k$: if $j \geq 0$ and $k \geq 2j+2$ then $F(j, k) = 2 \left( 2 - \frac{k+2}{(2j+(k+1))} \right)$. In particular, together with [9], this yields exact values for $F(0, k)$ for every $k$.

From [10] we easily deduce:

**Corollary 1.** Let $G$ be a planar graph, then $G$ is:

1. (0, 2)-colorable if $g(G) \geq 8$.
2. (0, 4)-colorable if $g(G) \geq 7$.
3. (1, 4)-colorable if $g(G) \geq 6$, and
4. (2, 6)-colorable if $g(G) \geq 5$.

Borodin et al. [5] constructed a planar graph with girth 6 which is not $(0, k)$-colorable for any $k$, and proved that every planar graph $G$ with $g(G) \geq 7$ is $(0, 8)$-colorable, and if $g(G) \geq 8$ then $G$ is $(0, 4)$-colorable. It follows from [6] that every planar graph $G$ with $g(G) \geq 7$ is $(1, 2)$-colorable, and every one with $g(G) \geq 6$ is $(1, 5)$-colorable. Borodin et al. [7] proved, among other results, that planar graphs with girth 5 are $(2, 13)$- and $(3, 7)$-colorable. Note that all these bounds are now strengthened by Corollary 1. Still, we suspect that **Corollary 1** can be further improved. Also, the result of Havet and Sereni [15] yields that every planar graph $G$ with $g(G) \geq 5$ (respectively, $g(G) \geq 6$, and $g(G) \geq 8$) is $(4, 4)$-choosable (respectively, $(2, 2)$-choosable, and $(1, 1)$-choosable).

The purpose of this paper is to prove Theorems 2 and 4.

**Theorem 2.** Every graph $G$ with $mad(G) \leq \frac{14}{5}$ is $(1, 1)$-colorable, and the restriction on $mad(G)$ is sharp.

**Corollary 3.** Every planar graph $G$ with $g(G) \geq 7$ is $(1, 1)$-colorable.

Note that **Theorem 2** and **Corollary 3** improve the above mentioned sufficient conditions for the $(1, 1)$-colorability due to Havet and Sereni [15]: $mad(G) \leq \frac{5}{3}$ for arbitrary graph $G$ and $g(G) \geq 8$ if $G$ is planar. Borodin and Ivanova [4] proved that every graph $G$ with $g(G) \geq 7$ and $mad(G) < \frac{14}{5}$ can be partitioned into two strong linear forests (each connected component of such forests is allowed to have at most two edges). Clearly, this result also follows from **Theorem 2**.

A. Pokrovskiy pointed out that **Theorem 2** has an application to sparse vertex Ramsey graphs. We say that $G \rightarrow (H_1, \ldots, H_k)$ if for every partition of the vertex set of $G$ into subsets $V_1, \ldots, V_k$ there exists $i$ such that $H_i$ is a subgraph of $G[V_i]$. Let $m_{cr}(H_1, \ldots, H_k) = \inf\{mad(F) : F(H_1, \ldots, H_k)\}$.

It is clear that a graph is $(d_1, \ldots, d_k)$-colorable if and only if $G \rightarrow (K_{d_1+1}, \ldots, K_{d_k+1})$. Furthermore, $F(j, k) = m_{cr}(K_{j+1}, K_{k+1})$. Borodin and Kostochka's results [9, 10] directly state exact values for $m_{cr}(K_{j+1}, K_{k+1})$ if $j \geq 2j+2$ or $j = 0$. Kurek
Sharpness of the restrictions in Theorems 2 and 4

ρ(graphs G with Theorem 4. Every graph G with (possiblesufficient condition for the This is the potential of S

u isomorphicto u otherneighborsof u 7

m (G) assuming the absence of special graphs, we will give structural results concerning a subgraph (1) we will describe all non-trivial subgraphsof a minimum counterexample with potential at most 3 as belonging to a finite set of special graphs, (2) assuming the absence of special graphs, we will give structural results concerning a subgraph G′ of a minimum counterexample with ρG′ ≥ 4, and (3) we conclude the proof with discharging. A similar method was used in [9,10]. In particular, an introduction to this method is [9], where the argument has fewer technicalities.

If G′ is a pendant block and w is the unique cut vertex in G′ of G, then w is the base of G′. A flag is a pendant block isomorphic to K4 − e, where the base is one of the vertices of degree 3 in K4 − e. A flag attached at a vertex u is a flag whose base has been glued to u. (In Fig. 1, we see flags attached at bases y1, y4, and y6.) The significance of a flag F in a graph G attached at u is that in each (1, 1)-coloring, there is a neighbor of u in F that is colored with the same color. Moreover, all other neighbors of u in G are colored with the other color. For the rest of the paper, we will assume that at most one flag is attached to each vertex. If two flags are attached to one vertex, then that subgraph is isomorphic to G1 from Section 2.

Unimportant vertices in a graph are (a) vertices of degree at most 1, (b) vertices of degree 2 contained in a triangle, (c) vertices of degree 3 contained in a flag. Semi-important vertices are vertices of degree 2 not contained in a triangle.

Fig. 1. A non-(1, 1)-colorable graph G3 with ρG3 = −1.
All other vertices are important. (In Fig. 1, we see unimportant vertices $u_i, v_i, w_i, z$, semi-important vertices $y_3, y_5, y_7$, and important vertices $y_1, y_2, y_4$, and $y_6$.)

We say that a graph $H$ is smaller than a graph $G$ (and denote this by $H < G$) if (i) $G$ has more important vertices than $H$, or (ii) $G$ and $H$ have the same number of important vertices and $G$ has more semi-important vertices than $H$, or (iii) $G$ and $H$ have the same amounts of important and semi-important vertices, and $|V(G)| > |V(H)|$, or (iv) $G$ has the same number of vertices in each class as $H$ and

$$
\sum_{u \in V(H)} d(u)^2 > \sum_{v \in V(G)} d(v)^2.
$$

Note that by this definition, if $H$ is a proper subgraph of $G$, then $H$ is smaller than $G$. Let $G$ be a counterexample to the main statement in Theorem 4 smallest with respect to the order above. In particular, $\rho_G \geq 0$. It is easy to see that $G$ is connected, $G$ has no separating edges, and hence $\delta(G) \geq 2$. A set $W \subseteq V(G)$ will be called an $i$-set if $\rho_G(W) = i$.

By a B-subgraph we mean the six-vertex subgraph of $G$ obtained from a flag and a triangle by gluing the base of the flag to a vertex of the triangle. The base and one other vertex in the triangle are considered as special. The point of this is that in each $(1, 1)$-coloring of $B$, each of the special vertices has neighbors of both colors, and they have distinct colors.

A super-flag $W$ is a $B$-subgraph of $G$ such that only special vertices of $W$ may have neighbors in $G$ outside of $W$. In Fig. 1, the subgraph induced by $(v_1, w_1, u_1, y_1, y_2, z)$ is a super-flag in which $y_1$ and $y_2$ are special. Recall once more that in each $(1, 1)$-coloring of a super-flag $W$, each of the special vertices has neighbors of both colors, and they have distinct colors.

Let $G'$ be a graph, $A \cup B = V(G')$, $\phi_A$ be a coloring of $G'[A]$, and $\phi_B$ be a coloring of $G'[B]$. If $\phi_A(u) = \phi_B(u)$ for all $u \in A \cap B$, then we define $\phi_A \cup \phi_B$ to be a coloring of $G'$ such that $\phi(w) = \phi_A(u)$ when $w \in A$ and $\phi(v) = \phi_B(v)$ otherwise.

**Lemma 5.** No super-flag is a separating set.

**Proof.** Suppose that a super-flag $W$ is a separating set. Let $V(G) - W = X \cup Y$ such that $E(X, Y) = \emptyset$.

By the minimality of $G$, let $f_X$ be a coloring of $X \cup W$ and $f_Y$ be a coloring of $Y \cup W$. Let $y_1$ and $y_2$ be the special vertices of $W$. By the symmetry of the coloring, we can assume that $f_X(y_1) = f_Y(y_1)$ and $f_X(y_2) = f_Y(y_2)$. Furthermore, if $v \in N(y_1) - W$ then $(f_X \cup f_Y)(v) \neq f(y_1)$ and if $u \in N(y_2) - W$ then $(f_X \cup f_Y)(u) \neq f(y_2)$.

Let $f$ be a coloring such that if $u \in X \cup W$ then $f(u) = f_X(u)$ and if $u \in Y \cup W$ otherwise. Then $f$ is a $(1, 1)$-coloring of $G'$.\hfill $\square$

**Lemma 6.** If $xy \in E(G)$, then $\max\{d(x), d(y)\} \geq 3$. Furthermore, if $xy$ is not part of a flag then $x$ or $y$ is important.

**Proof.** If $d(x) = d(y) = 2$, then let $g'$ be a $(1, 1)$-coloring on $G - x - y$. Let $ux, vy \in E(G)$. We create a $(1, 1)$-coloring $g$ on $G$ by setting $g_{|G - x - y} = g'$, $g(x) \neq g'(u)$, and $g(y) \neq g'(v)$. This is a contradiction.

Without loss of generality, assume that $d(x) \geq 3$. If $x$ is not important, then $d(x) = 3$ and $x$ is in a flag. But then $xy$ is a part of that flag.\hfill $\square$

Recall that the potential of a vertex set $S$ of a graph $G$ is $\rho_G(S) = 7|S| - 5|E(G[S])|$. It follows that $\rho_{K_3}(V(K_3)) = 7$, $\rho_{K_4}(V(K_4)) = 9$, $\rho_{K_5}(V(K_5)) = 6$, and $\rho_{K_6}(V(K_6)) = -2$. The minimum potential over all non-empty subsets of $V(G)$ is $\rho_G$. Let $G$ be a smallest graph, with respect to the order of graphs defined above, such that $\rho_G \geq 0$ and $G$ is not $(1, 1)$-colorable. By minimality, every proper subgraph of $G$ is $(1, 1)$-colorable.

**Fact 7.** Let $G'$ be a graph and $A, B, C \subseteq V(G')$ be such that $A \supseteq B$ and $A \cap C = \emptyset$. Then $\rho_G(A - B) = \rho_G(A) - \rho_G(B) + 5|E_G(A - B, B)|$ (equivalently, $\rho_G(A \cup C) = \rho_G(A) + \rho_G(C) - 5|E_G(A, C)|$).

We will use Fact 7 throughout the rest of the paper.

### 4. Sets with potential 2

**Proposition 8.** Let $\emptyset \neq T \subseteq V(G)$ be a set such that either (i) $\rho(T) \in \{0, 1\}$, or (ii) $\rho(T) = 2$ and for every $T' \subseteq T$, $\rho(T') \geq 3$. Let $F$ be a flag in $G$. Under these assumptions,

(a) $\delta(G[T]) \geq 2$.

(b) if $w \notin T$ then $|N(w) \cap T| \leq 1$.

(c) either $F \subseteq T$ or $F \cap T = \emptyset$, and

(d) every vertex $u$ in $T$ incident to an edge $uv$ leaving $T$ is important.

**Proof.** Because $\rho(T) \leq 6$, we have that $|T| \geq 2$.

If $u \in T$ is such that $|N(u) \cap T| \leq 1$, then $\rho(T - u) \leq \rho(T) - \rho(T - u) - 7 + 5$. If $\rho(T) \leq 1$, then this contradicts the assumption that $\rho_G \geq 0$. If $\rho(T) = 2$, then this contradicts that $\rho(T - v) \geq 3$ under assumption (ii). This proves (a).

Let $w \notin T$ with $|N(w) \cap T| \geq 2$. Then $\rho(T + w) \leq \rho(T) + 7 - 10 \leq -1$, which contradicts that $\rho_G \geq 0$. This proves (b).

If $|T \cap F| = 1$, then $T \cup F$ has three more vertices and at least five more edges than $T$. So in this case $\rho(T \cup F) \leq \rho(T) - 5 - 3 - 7 \leq -2 - 7 \leq -9$, which is a contradiction. Similarly, if $|T \cap F| = 2$, then $\rho(T \cup F) \leq \rho(T) - 5 - 2 - 2 \leq -9$, and if $|T \cap F| = 3$, then $\rho(T \cup F) \leq \rho(T) - 5 - 2 \leq -11$, which is a contradiction. This proves (c).

Let $uv \in E(G)$ and $u \in T$ and $v \notin T$. Due to (a), we have $d(u) \geq 3$. Therefore, if $u$ is important, then it must be in a flag. But if $u$ is in a flag $F$, then by (c), $F$ is contained in $T$, which implies that $d(u) \geq 4$.\hfill $\square$
Proposition 9. Let $B$ be a super-flag with special vertices $y_1$ and $y_2$. There is no $S \subseteq V(G)$ such that $\rho(S) \leq 2$ and $B \prec G[S]$.

Proof. By way of contradiction, let $S \subseteq V(G)$ be a set with potential at most 2. Let $f : S \rightarrow \{\alpha, \beta\}$ be a $(1, 1)$-coloring of $G[S]$. Let $S_\alpha$ be the set of vertices in $S$ colored $\alpha$, and $S_\beta$ be the set of vertices in $S$ colored $\beta$. Let $N_\alpha = N(S_\alpha) \cap (V(G) - S)$ and $N_\beta = N(S_\beta) \cap (V(G) - S)$. Let $G'$ be defined as follows:

$$V(G') = V(G) - S + V(B)$$

and

$$E(G') = E(G - S) + E(B) + \{uy_1 : u \in N_\alpha\} + \{vy_2 : v \in N_\beta\}.$$

Because $B \prec G[S]$ and the unimportant vertices of $G$ are unimportant in $G'$, it follows that $G' \prec G$. Suppose that there exists a $T \subseteq V(G')$ with $\rho(T) \leq -1$. Because such a $T$ was not in $G$, we have $T \cap B \neq \emptyset$. We may assume that $B \subseteq T$, because a quick examination of all possibilities will show that $\rho(G'(T) \cup B) \leq \rho(T')$. But then $\rho(G'(T - B) + S) \leq \rho(G'(T))$, which is a contradiction. Therefore $G'$ has a coloring $g'$.

By the symmetry of the coloring we may assume that $g'(y_1) = \alpha$. By the configuration of a super-flag, it follows that $g'(y_2) = \beta$. Moreover, it must be the case that $g'(u) = \beta$ for all $u \in N_\alpha$ and $g'(v) = \alpha$ for all $v \in N_\beta$. Therefore the coloring $g$ on $G$ defined by $g = g' \cup f$ is a $(1, 1)$-coloring. □

Proposition 10. Suppose that $\emptyset \neq S \subseteq V(G)$ with potential at most 2 and $x \in S$. Under these conditions $|S| \geq 5$ and $E(S - x, V(G) - S) \neq \emptyset$.

Proof. The smallest solution to the Diophantine relation $7|S| - 5|E(G[S])| \in \{0, 1, 2\}$ that is realized by a simple graph is $|S| = 5$, $|E(G[S])| = 7$.

By way of contradiction, let $S \subseteq V(G)$ have potential at most 2 and $E(S - x, V(G) - S) = \emptyset$. Let $G' = G - (S - x) + F$, where $F$ is a flag attached to $x$. Because $\delta(G) \geq 2$, and by Lemma 6, there must be at least two important vertices in $S$. Hence $G' \prec G$.

Suppose that there is a $T \subseteq V(G')$ with $\rho(G')(T) \leq -1$. Because $\rho(G') \geq 0$, it follows that $T \cap (F - x) \neq \emptyset$. We may assume that $F \subseteq T$, because a quick examination of all possibilities will show that $\rho(G'(T) \cup F) \leq \rho(G')(T)$. But then $\rho(G'(T - F) + (S - x)) \leq -1 + 4 + 2 - 7$, which contradicts $\rho(G) \geq 0$. By the minimality of $G$, we have that $G'$ has a $(1, 1)$-coloring $g'$.

Without loss of generality, assume that $g'(x) = \alpha$. By construction, it must be the case that for all $u \in N(x) \cap (V(G) - S)$, we have $g'(u) = \beta$. Let $f$ be a $(1, 1)$-coloring of $G[S]$ with $f(x) = \alpha$. Then $g = f \cup g'$ is a $(1, 1)$-coloring of $G$. □

Corollary 11. If $\emptyset \neq S \subseteq V(G)$ has potential at most 2, then $S$ contains exactly two important vertices and no semi-important vertices. Furthermore, $|S| \leq 6$.

Proof. Let $S' \subseteq S$ be a minimal non-empty set with potential at most 2. By Propositions 8.d and 10, $S'$ contains at least two important vertices. By Proposition 9, $S'$ contains at most two important vertices and four unimportant vertices. □

Proposition 12. There is no $S \subseteq V(G)$ such that $\rho(S) = 0$ and $|S| = 5$.

Proof. Suppose that such an $S$ exists. By Corollary 11, $S$ has important vertices $x_1, x_2$ and unimportant vertices $z_1, z_2, z_3$. If $G[S]$ contains a flag, then $|E(G[S])| \leq 6$ and $\rho(S) \geq 5$. Therefore each of $\{z_1, z_2, z_3\}$ has degree 2 and is in a triangle. By Lemma 6, the $z_i$’s are not adjacent to each other. Therefore $N(z_1) = N(z_2) = N(z_3) = \{x_1, x_2\}$ and $x_1 x_2 \in E(G)$.

Define $G' = G - S + F$, where the base of $F$ is $x$, and $N(v) = (N(x_1) \cup N(x_2)) - S$. If there exists a $T \subseteq V(G')$ with $\rho(G')(T) \leq -1$ then because $\rho(G') \geq 0$, it follows that $T \cap F \neq \emptyset$. We may assume that $F \subseteq T$, because a quick examination of all possibilities will show that $\rho(G'(T) \cup F) \leq \rho(G')(T)$. But then $\rho(G'(T - F) + S) \leq -1 + 3 + 0 \leq -1$, which is a contradiction. By construction, $G' \prec G$. By minimality of $G$, there is a coloring $g'$ of $G'$.

Without loss of generality, let $g'(v) = \alpha$. For all $u \in N(v) - F$, $g'(u) = \beta$. Create a coloring $g$ of $G$ as follows: Set $g|_{V(G) - S} = g'|_{V(G') - F}$, $g(x_1) = g(x_2) = \alpha$, and $g(z_1) = g(z_2) = g(z_3) = \beta$. This is a $(1, 1)$-coloring of $G$. □

Lemma 13. If $\emptyset \neq S \subseteq V(G)$ has potential at most 2, then $G[S]$ is a super-flag.

Proof. Let $\{y_1, y_2\}$ be the important vertices in $S$ described by Corollary 11.

By Corollary 11 and Proposition 12, we conclude that if $G[S]$ is not a super-flag then $|S| = 6$ and $S$ contains four vertices of degree 2 contained in a triangle. By Lemma 6, the neighborhood of each unimportant vertex is $\{y_1, y_2\}$ and $y_1 y_2 \in E(G)$. But then $G[S]$ contains nine edges and $\rho(S) \leq -3$, a contradiction. So $G[S]$ is a super-flag. □

As a corollary to Lemma 13, if $G' \subseteq G$, then $\rho(G') \geq 2$. Because adding one vertex and two edges reduces the potential by 3, we immediately have the following corollary.

Corollary 14. Let $\emptyset \neq T \subseteq V(G)$ and $x_1, x_2 \notin T$. Let $G' = G - T + x'$, where $N(x') = \{x_1, x_2\}$. If $x_1$ and $x_2$ are not in the same super-flag, then $\rho(G') \geq 0$. 

2642
5. Sets with potential 3

A 3-set is standard if it is a flag or has at least $|V(G)| - 1$ vertices. The goal of this section is to prove that all 3-sets of $G$ are standard. We will do this through a sequence of smaller statements on a nonstandard 3-set $X$ with fewest vertices in $G$. Let $X_0$ denote the set of vertices in $X$ that have neighbors outside of $X$.

**Proposition 15.** $G[X]$ is connected and $\delta(G[X]) \geq 2$.

**Proof.** If $G_1, \ldots, G_k$ are connected components of $G[X]$ and $k \geq 2$, then for some $i$, $\rho_c(V(G_i)) \leq \lfloor 3/k \rfloor \leq 1$, a contradiction to Lemma 13. If $x \in X$ and $d(x) \leq 1$, then $\rho_c(X - x) \leq 3 - 7 + 5 = 1$, a contradiction to Lemma 13.

**Proposition 16.** No vertex outside $X$ has more than one neighbor in $X$.

**Proof.** Suppose that $z \in V(G) - X$ has at least two neighbors in $X$. Let $X' = X + z$. Since $X$ is nonstandard, $X' \neq V(G)$, but $\rho(X') \leq \rho(X) + 7 - 2 \cdot 5 = 0$, a contradiction to Lemma 13.

**Corollary 17.** Each $x \in X_0$ is important.

**Proof.** By Proposition 15 and the definition of $X_0$, $d(x) \geq 3$. Suppose that $x$ is a 3-vertex in a flag $F$. Then exactly one vertex of $F$ is outside of $X$, and this vertex has at least two neighbors in $F$, a contradiction to Proposition 16.

**Proposition 18.** $G[X]$ has no $(1, 1)$-coloring in which all vertices in $X_0$ have the same color.

**Proof.** Suppose that $G[X]$ has a $(1, 1)$-coloring $f$ such that $f(x) = 1$ for each $x \in X_0$. Let $G'$ be obtained from $G - X$ by adding a new flag $F$ with base $x_0$ and adding an edge $z_0x_0$ for each $z \in V(G) - X$ that had a neighbor in $X$. Since the potential of $F$ is the same as that of $X$ in $G$, we have $\rho_c(A) \geq 0$ for every $A \subseteq V(G')$. By Lemma 6 and Proposition 15, $X$ had more than one important vertex. And any important vertex in $V(G') - x_0$ was important in $G$. So, $G'$ has a $(1, 1)$-coloring $g$. We may assume that $g(x_0) = 1$. Since $x_0$ has a neighbor of color 1 in $F$, it has no such neighbors in $V(G') - F$. So $g|_{V(G) - X} \cup f$ is a $(1, 1)$-coloring of $G$.

**Proposition 19.** Each $x \in X_0$ has a neighbor in $V(G)$ that is either important or semi-important.

**Proof.** Suppose that a vertex $z \in V(G) - X$ adjacent to $x \in X_0$ is unimportant. Since $\delta(G) \geq 2$, by Proposition 16 there is a $w \in N(x) \cap N(z) \cap (V(G) - X)$. By Lemma 6, $w$ is important unless $w$ is in a flag $F$. By Proposition 16, $F \cap X = \{x\}$. This is a contradiction, because $\rho(X + F) \leq 3 + 21 - 25 = -1$.

**Proposition 20.** $|X_0| \geq 3$.

**Proof.** By Proposition 18, $|X_0| \geq 2$ and if $X_0 = \{x_1, x_2\}$, then $f(x_1) \neq f(x_2)$ for every $(1, 1)$-coloring $f$ of $G[X]$. By definition, $\{x_1, x_2\}$ is a separating set. If $x_1$ and $x_2$ are the special vertices of some super-flag $H$ of $G$, then $H$ is also a separating set. But that contradicts Lemma 5. Therefore vertices $x_1$ and $x_2$ are not in the same super-flag of $G$.

Let $G_0$ be obtained from $G[X]$ by adding a vertex $y$ adjacent to $x_1$ and $x_2$. Adding a vertex of degree 2 to a subgraph decreases its potential by 3; therefore by Lemma 13 the potential of every subgraph of $G[X]$ is still non-negative. We claim that $G_0$ is smaller than $G$. Indeed, by Proposition 19, each of $x_1$ and $x_2$ has a neighbor in $V(G) - X$ that is not unimportant, and by Proposition 16 they are different. But $y$ is either semi-important or unimportant. Thus, $G_0$ is smaller than $G$ and hence has a $(1, 1)$-coloring $f$. By Proposition 18, $f(x_1) \neq f(x_2)$. By symmetry we may assume that $f(x_1) = f(y) = 1$ and $f(x_2) = 2$. Let $G_1$ be obtained from $G - (X - x_1 - x_2)$ by adding an edge $x_1x_2$ (if it is not in $E(G)$) and placing a flag $F$ on $x_2$. Note that the potential of $G_1[F]$ is 3 and that of $G_1[F + x_1]$ is $3 + 7 - 5 = 5$. So, $G_1$ has no sets of negative potential. Thus if $G_1$ is smaller than $G$, then it has a $(1, 1)$-coloring $g$. Since $x_2$ has a neighbor of its color in $F$, $g(x_1) \neq g(x_2)$. Therefore, renaming colors such that $g(x_1) = 1$ and $g(x_2) = 2$, we would have that $f|_{X_0} \cup g|_{G[X]}$ is a $(1, 1)$-coloring of $G$. Hence $G_1$ is not smaller than $G$. In this case, all vertices in $X - x_1 - x_2$ are unimportant and $|X| \leq 5$. Since no unimportant vertex can be an intermediate vertex in a shortest path and $G[X]$ is connected, $x_1x_2 \in E(G)$. So $\rho_c(\{x_1, x_2\}) = 14 - 5 = 9$. Adding a flag to a set decreases the potential by 4, adding a 2-vertex decreases it by 3, and adding two adjacent 2-vertices generates a subgraph forbidden by Lemma 6. Because $\rho_c(X) = 3$, it follows that $X - \{x_1, x_2\}$ is two 2-vertices adjacent to $x_1$ and $x_2$. But then $G[X]$ does have a $(1, 1)$-coloring $f$ with $f(x_1) = f(x_2)$, a contradiction to Proposition 18.

**Proposition 21.** In every $(1, 1)$-coloring of $G[X]$, each vertex in $X_0$ has neighbors of both colors.

**Proof.** Suppose that in $(1, 1)$-coloring $f$ of $G[X]$, a vertex $x \in X_0$ has no neighbors of color 1. Then by Proposition 15, $f(x) = 1$. By Proposition 19, $x$ has a neighbor $z \in V(G) - X$ that is not unimportant. Let $G'$ be obtained from $G - X$ as follows:

(a) add a flag $F$ attached to $z$;
(b) add a super-flag $Y$ with special vertices $y_1$ and $y_2$;
(c) for each $v \in V(G) - X - z$ that is adjacent to a vertex of color $i$ in $f$, join $v$ by an edge to $y_i$. 
By Proposition 16, there will be no confusion with (c).

Case 1: $G'$ has a $(1, 1)$-coloring $g$. We claim that $f \cup g$ is a $(1, 1)$-coloring. Indeed, if $g(z) = 2$, then this follows from the construction of $G'$. Moreover, if $g(z) = 1$, then because of $F$, $z$ has no neighbor of color 1 in $G' - (F - z)$. So, even though vertices $x$ and $z$ of color 1 are adjacent to each other, they do not have other neighbors of color 1 in $G$.

Case 2: $G'$ has no $(1, 1)$-colorings. By Proposition 20, $X$ contains at least three important vertices. So, since $z$ was not unimportant, $G'$ is smaller than $G$. Thus $G'$ has a set $Z$ with $\rho_G(Z) \leq -1$.

Case 2.1: $Z \cap Y \neq \emptyset$. The subgraph of a super-flag with smallest potential is the whole super-flag, so we may assume that $Y \subseteq Z$. If $z \notin Z$, then $F \cap Z = \emptyset$, and

$$\rho_G(X \cup (Z - Y)) \leq \rho_G(Z) - \rho_G(Y) + \rho_G(X) \leq -1 - 2 + 3 = 0,$$

a contradiction to Lemma 13. So $z \in Z$. Then $\rho_G(-y = z)(Z - (F - z)) \leq -1 - 21 + 25 = 3$. So, because of the edge $xz$,

$$\rho_G(X \cup (Z - (F - z) - Y)) \leq \rho_G(-y = z)(Z - (F - z)) - \rho_G(Y) + \rho_G(X) - 5|E(X, \{z\})| \leq 3 - 2 + 3 - 5 = -1,$$

which contradicts the assumption that $\rho_G \geq 0$.

Case 2.2: $Z \cap Y = \emptyset$. Then $z \in Z$. By the same calculation as in Case 2.1, $\rho_G(Z - (F - z)) \leq 3$ and

$$\rho_G(X \cup (Z - (F - z))) \leq \rho_G(X) + \rho_G(Z - (F - z)) - 5|E(X, Z - (F - z))| \leq 3 + 3 - 5 = 1.$$

By Lemma 13, we get $X \cup (Z - (F - z)) = V(G)$. But since there are at least two edges between $X$ and $V(G) - X$, we then have $\rho_G(V(G)) \leq 3 + 3 - 10 = -4$, which contradicts the assumption that $\rho_G \geq 0$. □

Proposition 22. For every $x, x' \in X_0$ such that $x$ and $x'$ are not in the same super-flag of $G$, there is a $(1, 1)$-coloring $f$ of $G[X]$ such that $f(x) \neq f(x')$.

Proof. Let $G''$ be obtained from $G[X]$ by adding a new vertex $v$ adjacent to $x$ and $x'$. By Corollary 14, $\rho_{G''} \geq 0$. By Proposition 19, $G''$ is smaller than $G$. So by the minimality of $G$, $G''$ has a $(1, 1)$-coloring $f$. By Proposition 21, both $x$ and $x'$ have neighbors of both colors in $G'' - v$. Thus $f(v) \neq f(x)$ and $f(v) \neq f(x')$. It follows that $f(x) = f(x')$. □

Lemma 23. Graph $G$ has no nonstandard 3-sets.

Proof. Suppose that $X$ is a minimum size nonstandard 3-set and $X_0$ is the set of vertices in $X$ adjacent to $V(G) - X$. Since $G[X]$ is smaller than $G$, it has a $(1, 1)$-coloring $f_1$. Let $X_1$ be the set of vertices $x \in X_0$ with $f_1(x) = 1$ and $X_2 = X_0 - X_1$. By changing the names of colors if needed, we may assume that $|X_1| \geq |X_2|$. Since $|X_0| \geq 3$, we have $|X_1| \geq 2$. By Proposition 18, $X_2 \neq \emptyset$.

Let $x, x' \in X_1$ and $y \in X_2$. Since each super-flag has only two important vertices, $y$ and $x$ are not in the same super-flag or $y$ and $x'$ are not in the same super-flag. So by Proposition 22, there is a $(1, 1)$-coloring $f_2$ of $G[X]$ such that $f_2(y) \in \{f_2(x), f_2(x')\}$. Thus, we have proved that

there is a $(1, 1)$-coloring $f_2$ of $G[X]$ distinct from $f_1$.

Let $Y_1 = \{x \in X_0 : f_1(z) = f_2(z)\}$ and $Y_2 = X_0 - Y_1$. By switching the names of the colors in $f_2$, we can achieve that

$$|Y_1| \geq |Y_2|.$$

Case 1: All vertices in $Y_2$ have the same color in $f_1$. Let $G_1$ be obtained from $G - X$ by

(a) adding a flag $F$ with base $y_0$,
(b) adding a copy $H$ of a super-flag disjoint from $F$ with special vertices $y_1$ and $y_2$,
(c) adding the edge $(z, y_0)$ for every $z \in V(G) - X$ that is adjacent to some $w \in Y_2$, and
(d) adding the edge $(z, y_{3-j})$ for every $z \in V(G) - X$ that is adjacent to some $w \in Y_1$ with $f_1(w) = j$, for $j = 1, 2$.

Suppose first that $G_1$ has a $(1, 1)$-coloring $g$. We may assume that $g(y_1) = 1$. Then $g \cup f_1$ or $g \cup f_2$ is a $(1, 1)$-coloring of $G$. So, $G_1$ has no $(1, 1)$-colorings. Suppose now that $\rho_{G_1}(Z) \leq -1$ for some $Z \subseteq V(G_1)$ and $Z$ has the smallest potential in $G_1$. Then $Z$ either contains $F$ or is disjoint from $F$, and similarly either contains $H$ or is disjoint from $H$. By the construction of $G_1$, it contains $F$ or $H$.

Case 1.1: $F \subset Z$ and $H \subset Z$. Then the potential of $(Z - F - H) \cup X$ in $G$ is at most

$$\rho_{G_1}(Z) - \rho_{G_1}(F) - \rho_{G_1}(H) + \rho_{G_1}(X) \leq -1 - 3 - 2 + 3 \leq -3,$$

which contradicts the assumption that $\rho_G \geq 0$.

Case 1.2: $F \subset Z$ and $H \cap Z = \emptyset$. Following the calculation of Case 1.1, the potential of $(Z - F) \cup X$ in $G$ is at least

$$-1 - 3 + 3 \leq -1$$

a contradiction.

Case 1.3: $F \cap Z = \emptyset$ and $H \subset Z$. Following the calculation of Case 1.1, the potential of $(Z - H) \cup X$ in $G$ is at most

$$\rho_{G_1}(Z) + \rho_{G_1}(H) + \rho_{G_1}(X) - 5|E(Y_2, V(G) - X)| \leq -1 - 2 + 3 - 5 = -5,$$

a contradiction.
Thus, $G_1$ satisfies the conditions of the theorem and has no $(1, 1)$-colorings. By the choice of $G$, it cannot be smaller than $G$. Let us check how this may happen. By Corollary 17, every vertex in $X_0$ is important. Since we have added to $V(G) - X$ at most three important vertices, we conclude that $X_0 = \{x, x', y\}$ and every other vertex of $X$ is unimportant. By Proposition 15, $G[X]$ is connected and because no unimportant vertex can be an intermediate vertex in a shortest $x, x'$-path (or $x, y$-path or $x', y$-path), it follows that $G[X_0]$ is connected. So, $G[X_0]$ has either two or three edges.

Case 1.A: $G[X_0]$ has three edges. Then $\rho(X_0) = 6$. Since adding a 2-vertex decreases the potential by 3 and adding a flag decreases it by 4, the only way to get $\rho(X) = 3$ is that we obtain $X$ by adding one 2-vertex. In this case, $G[X] = K_4 - e$. If we color the vertices of degree 3 in $G[X]$ with color 1, and the vertices of degree 2 in $G[X]$ with color 2, then the latter will have only neighbors of color 1, a contradiction to Proposition 21.

Case 1.B: $G[X_0]$ has two edges. This means that $G[X_0]$ is a path of length 2, say $(x_1, x_2, x_3)$, where $\{x_1, x_2, x_3\} = \{x, x', y\}$. Then $\rho(X_0) = 11$ and $\rho(X) - \rho(X_0) = -8$. Since the only way to express 8 as a sum of 3’s and 4’s is $8 = 4 + 4$, $G[X]$ is obtained from $G[X_0]$ by adding two flags. Then there is a $(1, 1)$-coloring of $G[X]$ such that $h(x_1) = h(x_3) \neq h(x_2)$. So, the vertex in $X_0$ not contained in a flag will have neighbors of only one color, a contradiction to Proposition 21.

Case 2: The set $Y_2$ has vertices of both colors in $f_1$. Then $2 \leq |Y_2| \leq |Y_1|$. The proof almost repeats the one for Case 1 with more case analysis at the end. Let $G_2$ be obtained from $G - X$ by

(a) adding two disjoint copies $H_1$ and $H_2$ of a super-flag with special vertices $y_{1,1}$ and $y_{1,2}$ in $H_1$ and $y_{2,1}$ and $y_{2,2}$ in $H_2$; and
(b) adding the edge $(z, y_{i,j})$ for every $z \in V(G) - X$ that is adjacent to some $w \in Y_i$ with $f_1(w) = j$, for all $i, j \in \{1, 2\}$.

Suppose first that $G_2$ has a $(1, 1)$-coloring $g$. We may assume that $g(y_{1,1}) = 1$. Then $g \cup f_1$ or $g \cup f_2$ is a $(1, 1)$-coloring of $G$. So, $G_2$ has no $(1, 1)$-colorings. Suppose now that $\rho_{G_2}(Z) \leq -1$ for some $Z \subseteq V(G_2)$ and $Z$ has the smallest potential in $G_2$. Then for $i = 1, 2$, $Z$ either contains $H_i$ or is disjoint from $H_i$. By construction, $Z$ contains $H_1$ or $H_2$.

Case 2.1: $H_1 \subseteq Z$ and $H_2 \subseteq Z$. Then the potential of $(Z - H_1 - H_2) \cup X$ in $G$ is at most

$$\rho_{G_2}(Z) - \rho_{G_2}(H_1) - \rho_{G_2}(H_2) + \rho_C(X) \leq -1 - 2 - 2 + 3 \leq -2,$$

which contradicts the assumption that $\rho_G \geq 0$.

Case 2.2: $H_1 \cap Z = \emptyset$ and $H_2 \subseteq Z$. Following the calculation of Case 2.1, the potential of $(Z - H_2) \cup X$ in $G$ is at most $-1 - 2 + 3 \leq 0$. If $(Z - H_2) \cup X \neq V(G)$, this contradicts Lemma 13. If $(Z - H_2) \cup X = V(G)$, then we did not take into account the contribution of edges connecting $Y_1$ with $V(G) - X$. So, in this case, the potential of $(Z - H_2) \cup X = V(G)$ is at most

$$\rho_{G_2}(Z) - \rho_{G_2}(H_2) + \rho_C(X) - 5|E(Y_1, V(G) - X)| \leq -1 - 2 + 3 - 5 \leq -5,$$

a contradiction.

Case 2.3: $H_2 \cap Z = \emptyset$ and $H_1 \subseteq Z$. The case is symmetric to Case 2.2.

Thus, $G_2$ satisfies the conditions of the theorem and has no $(1, 1)$-colorings. By the minimality of $G$, $G_2$ is not smaller than $G$. Let us check how this may happen. We have added to $V(G) - X$ only four important vertices. Using the same logic as at the end of Case 1, we conclude that $|X_0| = 4$, every other vertex of $X$ is unimportant, and $G[X_0]$ is connected. So, $G[X_0]$ has three, four or five edges.

Case 2.A: $G[X_0]$ has five edges. Then $\rho(X_0) = 3$ and hence $X = X_0$. In this case, $G[X] = G[X_0] = K_4 - e$. So we get a contradiction to Proposition 21 exactly as at the end of Case 1.A.

Case 2.B: $G[X_0]$ has four edges. Then $\rho(X_0) - \rho(X) = 5$ and there is no way to express 5 as a sum of 4’s and 3’s.

Case 2.C: $G[X_0]$ has three edges. Then $\rho(X_0) - \rho(X) = 10$. Since the only way to express 10 as a sum of 3’s and 4’s is $10 = 4 + 3 + 3$, $G[X]$ is obtained from $G[X_0]$ by adding one flag and two unimportant 2-vertices. Note that $G[X_0]$ is either $K_{1,3}$ or $P_4$. By Proposition 15, if $w$ is a leaf of $G[X_0]$, then $w$ belongs to a flag or is adjacent to an unimportant 2-vertex.

Case 2.C.1: $G[X_0] = K_{1,3}$. Let $x_0$ be the vertex of degree 3 in $G[X_0]$ and $x_1, x_2, x_3$ be the remaining vertices in $X_0$. By (3) we may, up to reordering of $x_1, x_2, x_3$, assume that $x_1$ belongs to a flag, one unimportant 2-vertex is adjacent to $x_0$ and $x_2$, and one unimportant 2-vertex is adjacent to $x_0$ and $x_2$. Then there is a $(1, 1)$-coloring of $G[X]$ such that all neighbors of $x_0$ have the same color, a contradiction to Proposition 18.

Case 2.C.2: $G[X_0] = P_4$. We may assume that this path is $(x_1, x_2, x_3, x_4)$. By symmetry, we may assume that the vertex in the flag is either $x_1$ or $x_3$. Suppose first that it is $x_3$. Then by (3), one unimportant 2-vertex, say $v_1$, is adjacent to $x_1$ and $x_3$, and the other unimportant 2-vertex, say $v_2$, is adjacent to $x_3$ and $x_4$. Then we let $f(x_1) = f(v_1) = f(x_3) = f(v_2) = 1$ and $f(x_3) = f(x_4) = 2$. In this coloring, both neighbors of $x_4$ have color 1, a contradiction to Proposition 21.

So, $x_1$ belongs to the flag. By (3), there is an unimportant 2-vertex $v_1$ adjacent to $x_3$ and $x_4$. Let $v_2$ be the other unimportant 2-vertex. Let $f(x_1) = f(x_3) = 1$ and $f(x_2) = f(x_4) = f(v_1) = f(v_2) = 2$. Then either this coloring extends to a $(1, 1)$-coloring of $G[X]$ (by coloring the vertices in the flag), or $v_2$ is adjacent to $x_3$ and $x_4$. In the former case, all neighbors of $x_3$ are colored with 2, a contradiction to Proposition 21. In the latter case, we recolor $v_2$ with 1, and get a $(1, 1)$-coloring of $G[X]$ in which both neighbors of $x_2$ are colored with 1, again a contradiction to Proposition 21.

Attaching a flag to a vertex reduces the potential by 4. Therefore we immediately get the following corollary from Lemmas 13 and 23.

**Corollary 24.** Let $T \subseteq V(G)$ and $w \notin T$. Let $G' = G - T + F$, where $F$ is a flag attached at $w$. If $|T| \geq 2$ and $w$ is not in a flag or super-flag, then $\rho_{G'} \geq 0$. 


6. Reducible configurations

For every super-flag, exactly one of the two special vertices is in a flag. We will call the special vertex not in a flag the secondary base of the super-flag. We will show in Lemma 26 that every secondary base has degree at least 4.

Lemma 25. Let each of \( u \) and \( v \) be a base of a flag or a secondary base of a super-flag. If \( u \) and \( v \) are not special vertices of the same super-flag, then the distance between \( u \) and \( v \) is at least 3.

Proof. Let \( A_1 \) be the flag or super-flag attached at \( u \) and \( A_2 \) be the flag or super-flag attached at \( v \). Let \( P \) be a shortest path between \( u \) and \( v \). If \( |V(P)| < 3 \) then \( \rho_G(A_1 + P + A_2) \leq 3 + 3 - 3 \). Because \( A_1 + P + A_2 \) is neither a super-flag nor a flag, and by Lemmas 13 and 23, \( A_1 + P + A_2 = V(G) \). But then \( G \) has a \((1, 1)\)-coloring. \( \square \)

Lemma 26. Let \( v \in V(G) \) be a vertex with degree 3. If \( v \) is adjacent to a vertex of degree 2, then \( v \) is in a flag.

Proof. By way of contradiction, let \( N(v) = \{x, y, z\} \) and \( v \) not be in a flag. Note that \( v \) is important and every neighbor of \( v \) with degree at least 3 is important.

Case 1: \( v \) is in a super-flag. If \( v \) is in a super-flag, then \( G \) has degree 3, and is not in a flag, then \( v \) is the secondary base of that super-flag. Without loss of generality, let \( x \) be outside of that super-flag, \( z \) be the other special vertex of the super-flag, and \( N(y) = \{v, z\} \). Let \( F \) be the flag attached at \( z \).

Case 1.1: \( d(y) \geq 3 \). Let \( G' = G - y + y' \), where \( N(y') = \{x, z\} \). Note that \( y \) is important in \( G \), but \( v \) and \( y' \) are at most semi-important in \( G' \). Therefore \( G' \not\subset G \). By Corollary 14, \( \rho_{G'} \geq 0 \). Therefore we may find a function \( g' : G' \to \{\alpha, \beta\} \) such that \( g' \) is a \((1, 1)\)-coloring of \( G' \).

Without loss of generality, assume that \( g'(z) = \alpha \). Because \( F \) is attached to \( y \) in \( G' \), it follows that \( g'(u) = g'(y') = \beta \). From this, we deduce that \( g'(x) = \alpha \). We may generate a \((1, 1)\)-coloring \( g \) of \( G \) by setting \( g|_{V(G)-y} = g'_|{V(G)-y'} \) and \( g(y') = \beta \).

Case 1.2: \( N(x) = \{v, a\} \). By Lemma 6, \( a \) is important. Let \( S = F + v + y' \); in other words, \( G[S] \) is the super-flag containing \( v \). Let \( G = G - \{v, z, y\} + F', \) where \( F' \) is a flag attached at \( a \). Because \( v \) and \( a \) were important in \( G \), \( G' \not\subset G \). By Lemma 25, \( a \) is not in a flag or super-flag. So by Corollary 24, \( \rho_{G'} \geq 0 \). By the minimality of \( G' \), there exists a \((1, 1)\)-coloring \( g' \) of \( G' \). We create a \((1, 1)\)-coloring \( g \) of \( G \) by setting \( g|_{G-v} = g'|_{G'-v} \), \( \alpha \) and \( \beta \) such that \( g \) is a \((1, 1)\)-coloring of \( G' \).

Without loss of generality, assume \( g'(v) = \alpha \). Let \( g \) be a coloring of \( G \) where \( g|_{G-v-x-y-v} = g'|_{G'-v} \).

- If \( g'(z) = \beta \), then color \( g(x) = \beta \), \( g(v) = \alpha \), and \( g(y') \neq g(b) \).
- If \( g'(z) = \alpha \) and \( g'(v) = \beta \) then either \( g'(u) = \alpha \) or \( g'(y') = \alpha \). Furthermore, for all \( u \in N_{G'}(a) - \{x, y'\} \), \( g'(u) = \beta \). Color \( g(x) = \alpha \), \( g(v) = \beta \), and \( g(y') \neq g(b) \).
- If \( g'(z) = g'(v) = \alpha \) then color \( g(x) = g(y') = g(x) = g(v) = \alpha \).
- If \( g'(z) = g'(v) = \alpha \) and \( g'(b) = \beta \), then color \( g(x) = g(v) = \beta \) and \( g(y) = \alpha \).

The above assumptions exhaust all possibilities for \( g' \). Moreover, each provides a \((1, 1)\)-coloring of \( G \).

Case 2.1: \( a \neq b \) and \( b \neq z \). Let \( G' = G-\{y+y'+y\} \), where \( y \) is a flag attached at \( z \). If \( z \) was in a flag \( F' \) in \( G \), then \( F' \) \( b \) forms a super-flag containing \( v \), which contradicts the assumption that \( v \) is not in a super-flag. So by Corollary 24, \( \rho_{G'} \geq 0 \).

- If \( v \) and \( z \) are important in \( G \), we have \( G' \not\subset G \).

By minimality of \( G \), there exists a \((1, 1)\)-coloring \( g' \) of \( G' \). Note that if \( w \in N_{G'}(z) - F \) then \( g'(w) \neq g'(z) \). Let \( g \) be a coloring of \( G \) where \( g|_{G-v} = g'|_{G'-(v-z)} \), \( g(x) = g(y) \neq g'(a) \), and \( g(v) = g'(a) \). Either \( g(z) = g(v) \) or \( g(z) = g(y) \), but not both. Therefore there is only one neighbor with the same color for each vertex in \( g \).

Case 2.2: \( a = b \). Since \( v \) is not in a flag, \( G' = G-y+y' \), where \( N(y') = \{v, z\} \). Because \( y \) was half important and \( y' \) is unimportant, \( G' \) is smaller than \( G \). By Corollary 14, \( \rho_{G'} \geq 0 \). By minimality of \( G \), there exists a \((1, 1)\)-coloring \( g' : V(G') \to \{\alpha, \beta\} \).

Without loss of generality, let \( g'(z) = \alpha \). Thus \( g'(x) = \alpha \) or \( \alpha \in \{g'(v), g'(y')\} \). If neither of these statements is true then \( g'(v) = g'(x) = g'(y') = \beta \), and \( v \) is adjacent to two vertices with the same color in \( g' \).

Let \( g \) be a coloring on \( G \) where \( g|_{G-v-x-y-v} = g'|_{G'-(v-z)} \) and

- if \( g'(x) = \alpha \) then \( g(x) = \alpha \), \( g(y) \neq g(a) \), and \( g(v) = \beta \), or
- otherwise if \( \alpha \in \{g'(v), g'(y')\} \), then \( g(x) = g(y) \neq g'(a) \), and \( g(v) = g'(a) \).
Then $g$ is a $(1, 1)$-coloring.

Case 3: $x$ is adjacent to exactly one vertex of degree $2$. Let $N(x) = \{v, a\}$.

Case 3.1: $a \not\in \{y, z\}$. Then $x$ is semi-important. Let $G' = G - x + x'$, where $N(x') = \{y, z\}$. Because $x'$ and $v$ are at most semi-important in $G'$, it follows that $G'$ is smaller than $G$. If $y$ and $z$ are in the same super-flag $Y$, then $\rho_G(Y + v) \leq -1$, which is a contradiction. So by Corollary 24, $\rho_G \geq 0$. By minimality of $G$, there exists a $(1, 1)$-coloring $g'$ of $G'$. Without loss of generality, let $g'(x) = \alpha$.

If $g'(z) = \alpha$, then create a coloring $g|_{V(G) - x - v} = g'|_{V(G) - x - v}, g(v) = \beta$, and $g(x) \neq g'(a)$. This is a $(1, 1)$-coloring of $G$.

So we may assume that $g'(z) = \beta$. Because $g'$ is a $(1, 1)$-coloring, it follows that $g'(v) \neq g'(x')$, or else $z$ or $x$ will have two neighbors with the same color. Without loss of generality we may assume that $g'(v) = \beta$ and $g'(x') = \alpha$. Therefore all other neighbors of $z$ have color $\beta$ and all other vertices of $y$ have color $\alpha$. We color $G$ with coloring $g|_{V(G) - x - v} = g'|_{V(G) - x - v}, g(x) \neq g'(a)$, and $g(v) = g'(a)$. Note that $g(v)$ may be the same as $g(y)$ or $g(z)$, but it will not be the same as both. Hence, $g$ is a $(1, 1)$-coloring of $G$.

Case 3.2: $a = y$. Let $G' = G - x - v + F$, where $F$ is a flag attached at $y$. Then $G' \prec G$. If $y$ was in a flag $F'$ in $G$, then $F', v, x$ form a super-flag containing $v$, which contradicts the assumption that $v$ is not in a super-flag. So by Corollary 24, $\rho_G \geq 0$. By minimality of $G$, there exists a $(1, 1)$-coloring $g'$ of $G'$.

Note that if $w \in N_{G'}(y) - F$ then $g'(w) \neq g'(y)$. Let $g$ be a coloring of $G$ where $g|_{V(G) - x - v} = g'|_{V(G) - (F - y)}, g(x) = g'(z)$, and $g(v) \neq g'(z)$. Either $g(y) = g'(v)$ or $g(y) = g'(x)$, but not both. Therefore each vertex in $G$ has at most one neighbor with the same color in $g$, which is a contradiction because $G$ is not $(1, 1)$-colorable.

Case 4: $v$ is adjacent to three vertices of degree $2$. Let $G' = G - v$. Since $G'$ is a proper subgraph, it has a coloring $g'$. In $G'$, $x, y$, and $z$ all have degree $1$. Without loss of generality, we may assume that each of them has no neighbors with the same color as themselves. Extend the coloring on $G'$ to a coloring on $G$ by coloring $v$ the color that appears the least often in the list $(g'(x), g'(y), g'(z))$. Because some color appears at most once and that vertex has no other neighbors with the same color, this is a $(1, 1)$-coloring. $\square$

Corollary 27. If $y$ is a secondary base of a super-flag in $G$, then $d(y) \geq 4$.

Proof. By construction, each special vertex of a super-flag is adjacent to a vertex of degree $2$. By Lemma 6, $d(y) \geq 3$. By Lemma 26, $d(y) \neq 3$. $\square$

Lemma 28. Let $F$ be a flag in $G$. Then $|E(F, V(G) - F)| \geq 3$.

Clearly $|E(F, V(G) - F)| \geq 2$, or else $G$ would have a separating edge. In order to prove the above lemma, we will need the following result.

Proposition 29. If $F$ is a flag in $G$ with base $v$ and $N(v) - F = \{x, y\}$, then both $x$ and $y$ are important.

Proof. By way of contradiction, assume that $x$ is semi-important or unimportant. By Lemma 25, $d(x) \leq 2$. If $x$ is in a triangle, then $v$ and $y$ are the special vertices of a super-flag and Proposition 10 is contradicted. Therefore $x$ is semi-important and $N(x) = \{x, a\}$. Let $G' = G - F - x + F'$, where $F'$ is a flag attached at $a$.

By Lemma 25, $a$ is not in a flag of the super-flag. So by Corollary 24, $\rho_G \geq 0$. In $G$, $v$ is important and $x$ is semi-important. No vertex is more important in $G'$ than in $G$, so $G' \prec G$. By minimality of $G$, there exists a $(1, 1)$-coloring $g'$ of $G'$.

We have $g'(a) \neq g'(z)$ for all $z \in N_{G'}(a) - F'$. Then $G$ has a $(1, 1)$-coloring $g$, where $g|_{G - F - x} = g'|_{G' - (F' - a)}, g(v) \neq g'(y)$ and $g(x) = g'(y)$. $\square$

Proof of Lemma 28. Let $u$ be in flag $F$ and $N(u) - F = \{x, y\}$.

Case 1: $xy \not\in E(G)$ and $N(x) \cap N(y) = \{v\}$. Let $G' = G - F - x + y + z$, where $N(z) = (N(y) \cup N(x)) - v$. If $u \in V(G') - z$ is important in $G'$, then $u$ is important in $G$. Because both $x$ and $y$ are important in $G$, we have $G' \prec G$.

If $T \subset V(G')$ such that $\rho_G(T) \leq -1$, then $z \in T$. It follows that

$$\rho_G(T - z + x + y + F) = \rho_G(T) + 7(6 - 1) - 5(7) - \rho_G(T) \leq -1,$$

which contradicts the assumption that $\rho_G \geq 0$.

Therefore $G'$ has a $(1, 1)$-coloring, $g'$. We can create a $(1, 1)$-coloring of $g$ by setting $g|_{G - F - x - y} = g'|_{G' - z}, g(x) = g(y) = g'(z)$, and $g(v) \neq g'(z)$. $\square$

Case 2: $xy \in E(G)$ or there exists a $w$ such that $w \in (N(x) \cap N(y)) - v$. Let $G' = G - F - x + y + z + F'$, where $N(z) = (N(y) \cup N(x)) - v$ and $F'$ is a flag attached at $z$. If $u \in V(G') - z$ is important in $G'$, then $u$ is important in $G$. Because both $x$ and $y$ are important in $G$, we have $G' \prec G$.

If $T \subset V(G')$ such that $\rho_G(T) \leq -1$, then $z \in T$. Because of edge $xy$ or because $\|[ux, uy]\| = \|[uwz]\| + 1$, we get one extra edge over Case 1, and so

$$\rho_G(T - F' + x + y + F) \leq \rho_G(T) + 7(6 - 4) - 5(7 + 1 - 5) \leq -2,$$

which contradicts the assumption that $\rho_G \geq 0$.

Therefore $G'$ has a $(1, 1)$-coloring, $g'$. Furthermore, for all $u \in N_{G'}(z) - F', g'(u) \neq g'(z)$. We can create a $(1, 1)$-coloring of $g$ by setting $g|_{G - F - x - y} = g'|_{G' - F'}, g(x) = g(y) = g'(z)$, and $g(v) \neq g'(z)$. $\square$
Lemma 30. If \( v \in V(G) \) such that \( N(v) = \{u_1, u_2, u_3, u_4\} \) and \( N(u_i) = \{x_i, v\} \) for all \( i \), then \( x_i \) is in a flag or a super-flag for all \( i \).

Proof. Without loss of generality, let \( v, u_i \) and \( x_i \) be as above, and assume that \( x_i \) is not in a flag or a super-flag. By Lemma 6, each \( x_i \) is important. Let \( G' = G - \{v, u_1, u_2, u_3, u_4\} + F \), where \( F \) is a flag attached at \( x_1 \). By construction, \( G' \prec G \). By Corollary 24, \( \rho_G \geq 0 \). By minimality of \( G \), there is a \((1, 1)\)-coloring \( g' \) of \( G' \).

Because of the flag \( F \), for all \( w \in N_G(x_1) - u_1 \), we have \( g'(w) \neq g'(x_1) \). We construct a \((1, 1)\)-coloring \( g \) of \( G \) as follows:

- Set \( g|_{G - \{v, u_1, u_2, u_3, u_4\}} = g'|_{G - \{v, u_1, u_2, u_3, u_4\}} \).
- Set \( g(u_i) \neq g'(x_i) \) for \( i \in \{2, 3, 4\} \).
- Set \( g(v) \) equal to the color that appears the least in the list \( g(u_2), g(u_3), g(u_4) \).
- Set \( g(u_1) \neq g(v) \).

Then \( g \) is a \((1, 1)\)-coloring of \( G \), which is a contradiction. \( \square \)

7. Proof of the theorem

By assumption, on \( G \), we have

\[
\sum_{v \in V(G)} (5d(v) - 14) \leq 0.
\]  

The initial charge of each vertex \( v \) of \( G \) is \( \mu(v) = 5d(v) - 14 \), and the final charge \( \mu^*(v) \) is determined by applying the following rules:

R1. Every 2-vertex in a flag gets charge 4 from the base of the flag.
R2. Let \( x \) be the base of a flag \( F \) or the secondary base of a super-flag \( H \). Every vertex adjacent to \( x \) outside of \( F \) or \( H \) gets charge 2.5 from \( x \).
R3. Every 2-vertex \( u \) adjacent to a base \( x \) of a flag \( F \), where \( u \) is not in \( F \), gets from its neighbor charge 1.5.
R4. Every 2-vertex \( u \) adjacent to a secondary base \( x \) of a super-flag \( H \), where \( u \) is not in \( H \), gets from its other neighbor charge 1.5.
R5. Every 2-vertex not adjacent to a base of a flag or a secondary base of a super-flag gets charge 2 from each of its neighbors.

By Lemma 25, the application of Rules 2–4 is well-defined. By Lemma 6, the application of Rule 5 is well-defined.

Lemma 31. For every \( v \in V(G) \), \( \mu^*(v) \geq 0 \). Moreover, if \( d(v) \notin \{2, 4\} \), then \( \mu^*(v) > 0 \).

Proof. Recall that \( \delta(G) \geq 2 \). Note that by Lemma 28, each base of a flag has degree at least 6. If \( x \) is the base of a flag \( F \), then by R1, it gives charge 8 to the two 2-vertices in \( F \) and charge 2.5(d(x) - 3) to the neighbors outside of \( F \). So

\[
\mu^*(x) = 5d(x) - 14 - 2.5(d(x) - 3) = 2.5d(x) - 14.5 \geq 2.5 \cdot 6 - 14.5 = 0.5.
\]

A 3-vertex \( v \) in a flag does nothing to any other vertex, so \( \mu^*(v) = \mu(v) = 5d(v) - 14 = 1 \). A 2-vertex \( v \) in a flag receives charge 4 from the base of the flag, so \( \mu^*(v) = 5 \cdot 2 - 14 + 4 = 0 \).

If \( w \) is the 2-vertex in a super-flag \( H \) that is not in a flag, then \( w \) gets charge 2.5 from the base of the flag in \( H \), because \( w \) is not in the flag. Furthermore, \( w \) gets charge 1.5 from the secondary base of \( H \), because \( w \) is in the same super-flag and hence R3 applies and not R2. So

\[
\mu^*(w) = 5d(w) - 14 + 2.5 + 1.5 = 0.
\]

Let \( y \) be a secondary base for a super-flag \( H \). By R2, \( y \) receives 2.5 charge from the other special vertex \( y' \) of \( H \) (because \( y' \) is the base of a flag). By R3, \( y \) sends 1.5 charge to its other neighbor in \( H \) (see the discussion immediately above). By Corollary 27, every secondary base has degree at least 4. So

\[
\mu^*(y) \geq 5d(y) - 14 - 1.5 + 2.5 - 2.5(d(y) - 2) = 2.5d(y) - 8 \geq 2.
\]

For vertices not in flags or super-flags, we consider cases according to their degrees.

Case 1: \( d(v) = 2 \). Since by Lemma 6, \( v \) has no neighbors of degree 2, by Rules R2–R5, \( v \) gets from its neighbors the total charge at least 4, so \( \mu^*(v) \geq 5 - 2 - 14 + 4 = 0 \).

Case 2: \( d(v) = 3 \). Since \( v \) is not in a flag, by Lemma 26, \( v \) has no adjacent 2-vertices, so \( \mu^*(v) \geq \mu(v) = 5d(v) - 14 = 1 \).

Case 3: \( d(v) \geq 5 \). Since \( v \) is not in a flag, it gives at most 2 to each neighbor. So,

\[
\mu^*(v) = 5d(v) - 14 - 2d(v) = 3d(v) - 14 \geq 3 \cdot 5 - 14 = 1.
\]

Case 4: \( d(v) = 4 \). Since \( v \) is not in a flag, it gives at most 2 charge to each neighbor of degree 2. So, if \( v \) has at most three neighbors of degree 2, then \( \mu^*(v) \geq 5 \cdot 4 - 14 - 2 \cdot 3 = 0 \). Suppose that all neighbors of \( v \) are 2-vertices. Then by Lemma 30, each of these neighbors has a neighbor in a flag or a super-flag. So, by Rules R3 and R4, \( v \) sends to each neighbor only 1.5 charge. Thus, in this case, \( \mu^*(v) = 5 \cdot 4 - 14 - 1.5 \cdot 4 = 0 \). \( \square \)
Remark 1. By the proof of Case 4, if \( d(v) = 4 \) and \( \mu^*(v) = 0 \), then \( v \) has at least three neighbors of degree 2.

By the above lemma, in order for (4) to hold, we need \( \mu^*(v) = 0 \) for every \( v \in V(G) \). Then by the same lemma, \( G \) has only vertices of degree 2 and 4. By Remark 1, each 4-vertex has at most one neighbor of degree 4, and by Lemma 6, each 2-vertex has no neighbors of degree 2. In such a graph \( G \), if we color all 4-vertices with color 1 and all 2-vertices with color 2, then we get a \((1,0)\)-coloring of \( G \), a contradiction.

Acknowledgments

The authors are indebted to Anna Ivanova for her help in improving the presentation and to Alexey Pokrovskiy for pointing out an application to vertex Ramsey graphs. We are also grateful for the helpful comments by the anonymous referees.

The research of the first author was supported in part by grants 08-01-00673 and 09-01-00244 of the Russian Foundation for Basic Research.

The research of the second author was supported in part by NSF grant DMS-0965587 and by the Ministry of Education and Science of the Russian Federation (Contract No. 14.740.11.0868).

The research of the third author was partially supported by the Arnold O. Beckman Research Award of the University of Illinois at Urbana-Champaign.

References