Extremal problems on Δ-systems

A.V.Kostochka
Institute of Mathematics
Siberian Branch
Russian Academy of Sciences *

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Abstract

A family of sets is called a Δ-*system* (respectively, a *weak* Δ-*system*) if the intersection of any two sets is the same (respectively, the cardinality of the intersection of any two sets is the same). In 1960, P.Erdős and R.Rado started studying the maximum size of a k-uniform hypergraph not containing a Δ-system of a given size. The aim of the present article is to survey the progress and state of art in this and related problems.

1 Introduction

In connection with some problems in Number Theory, P.Erdős and R.Rado [12] introduced the notion of a Δ-system. They called a family $\mathcal{H}$ of sets a Δ-*system* if every two members of $\mathcal{H}$ have the same intersection. Define $f(k, r)$ to be the least cardinal so that any k-uniform family of more than $f(k, r)$ sets contains a Δ-system consisting of r sets. Erdős and Rado [12, 13] completely determined $f(k, r)$ in case at least one of k and r is infinite and found some upper and lower bounds for the case that both k and r are finite.

In 1974, Erdős, E.Milner and Rado [11] introduced the related notion of a weak Δ-system. A *weak* Δ-*system* is a family of sets where all pairs of sets have the same intersection size. Let $g(k, r)$ be the least cardinal so that every k-uniform family of more than $g(k, r)$ sets contains a weak Δ-system consisting of r sets. Erdős, Milner and Rado [11] found the values of $g(k, r)$ in case of infinite k and r assuming the generalized continuum hypothesis.

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Similar problems for families having a fixed cardinality of the ground set were introduced in 1978 by Erdős and E. Szemerédi [14]. They defined $F(n,r)$ to be the largest integer so that there exists a family $\mathcal{F}$ of subsets of an n-element set which does not contain a Δ-system of r sets and $G(n,r)$ to be the largest integer so that there exists a family $\mathcal{F}$ of subsets of an n-element set which does not contain a weak Δ-system of r sets.

The problems of estimating $f(k,r)$, $g(k,r)$, $F(n,r)$ and $G(n,r)$ have been attracting attention of many Mathematicians and were among favorite problems of Erdős for decades.

In this article, we survey the progress in studying these four functions, each of the subsequent sections devoted to a function. We focus the attention more on constructions than on proofs.

2 The original problem

The first and most famous problem is about $f(k,r)$. Erdős and Rado [12] proved that

$$(r-1)^k \le f(k,r) \le (r-1)^k k! \left\{1 - \sum_{t=1}^{k-1} \frac{t}{(t+1)!(r-1)^t}\right\}. \tag{1}$$

The construction providing the lower bound is as follows.

Construction 1. Let $X_1,\dots,X_k$ be disjoint sets of cardinality $r-1$ each. Let $\mathcal{F} = \{(x_1,\dots,x_k) \mid x_i \in X_i,\ i=1,\dots,k\}$. Clearly, $|\mathcal{F}| = (r-1)^k$. Suppose that some members $A_1,\dots,A_r$ of $\mathcal{F}$ form a Δ-system. Since these sets are distinct, there is an element x which belongs to exactly one of $A_1,\dots,A_r$. We may assume that $x \in A_1 \cap X_1$. Then all the r sets $A_i \cap X_1$, $i=1,\dots,r$, (each consisting of a single element) must be disjoint. Since $|X_1| = r-1$, this is impossible.

Erdős and Rado [12] also conjectured that for each r, there exists a constant C_r so that $f(k,r) < C_r^k$. Erdős (see [9]) has offered 1000 dollars for the proof or disproof of this for $r=3$.

The next remarkable paper in this direction was that of L. Abbott, D. Hanson, and N. Sauer [5]. They completely solved the case $k=2$ (namely, they showed that $f(2,r) = r(r-1)$ for odd r and $f(2,r) = r(r-1.5)$ for even r), improved the upper bound in (1) to $(k+1)!\left(\frac{r-1+\sqrt{r^2+6r-7}}{4}\right)^k$ and the lower bound for $f(k,3)$ to $2\cdot 10^{k/2-c\log k}$. This is still the best known lower bound. It is derived from their construction for every positive integer t of an intersecting 3^t-uniform family $\mathcal{F}_t$ of cardinality $10^{(3^t-1)/2}$ not containing a Δ-system of size 3. A description of the construction is as follows.

Construction 2. We use induction on t. It is a routine to check that the family $\mathcal{F}_1 = \{\{1,2,3\},\{1,2,4\},\{1,3,5\},\{1,4,6\},\{1,5,6\},\{2,3,6\},\{2,4,5\},\{2,5,6\},$ $\{3,4,5\},\{3,4,6\}\}$ with the ground set $\{1,\dots,6\}$ is what we need for $t=1$. Suppose that we have constructed an intersecting 3^{t-1}-uniform family $\mathcal{F}_{t-1}$ with a ground set

V of cardinality $10^{(3^{t-1}-1)/2}$ not containing a Δ-system of size 3. Let $\mathcal{F}_t$ have the ground set $V_1 \cup \ldots \cup V_6$, where every V_i is a copy of V; the members of $\mathcal{F}_t$ are the sets of the kind $E_\alpha \cup E_\beta \cup E_\gamma$, where $\{\alpha, \beta, \gamma\}$ is an edge in $\mathcal{F}_1$ and E_α, E_β and E_γ are arbitrary members of copies of $\mathcal{F}_{t-1}$ on the sets V_α, V_β and V_γ, respectively. Then

$$|\mathcal{F}_t| = |\mathcal{F}_1| \cdot |\mathcal{F}_{t-1}|^3 = 10 \cdot 10^{3(3^{t-1}-1)/2} = 10^{(3^t-1)/2}.$$

Since $\mathcal{F}_{t-1}$ and $\mathcal{F}_1$ both are intersecting families, $\mathcal{F}_t$ also is an intersecting family. To see that $\mathcal{F}_t$ does not contain a Δ-system of size 3, consider three arbitrary members A, B and C of $\mathcal{F}_t$.

CASE 1. The set $A \cup B \cup C$ meets at least four sets V_i. Then, due to construction of $\mathcal{F}_1$, some V_j (say, V_1) meets exactly two of A, B and C, say, A and B. Since $\mathcal{F}_{t-1}$ is an intersecting family, there exists some $v \in A \cap B \cap V_1$. This v witnesses that A, B and C do not form a Δ-system.

CASE 2. Every of A, B and C meets the same three sets V_i, say, V_1, V_2 and V_3. Since A, B and C are distinct sets, we may suppose that they do not coincide on V_1. Then, due to the properties of $\mathcal{F}_{t-1}$, some element w of V_1 belongs to exactly two of A, B and C. This w witnesses that A, B and C do not form a Δ-system.

It would be very interesting to improve the construction even just a bit. But maybe it is optimal.

The next upper bounds on $f(k, r)$ are due to J. H. Spencer [20]. He proved that for every fixed r and any $\epsilon > 0$,

$$f(k, r) < C(1+\epsilon)^k k!$$

and that

$$f(k, 3) < e^{ck^{3/4}} k!\ .$$

Z. Fűredi and J. Kahn (see [10]) proved that $f(k, 3) < e^{c\sqrt{k}} k!$. Currently best upper bound on $f(k, r)$ for small r is the following [16]:

For each integers $r > 2$ and $\alpha > 1$, there exists $D(r, \alpha)$ such that for all k,

$$f(k, r) \leq D(r, \alpha) k! \left(\frac{(\log \log \log k)^2}{\alpha \log \log k} \right)^k . \tag{2}$$

This bound is less than $k!$ but not much less and the gap between lower and upper bounds is still drastic.

A better situation takes place for large r and small k. As was mentioned above, Abbott, Hanson, and Sauer [5] completely solved the case $k = 2$. Then Abbott and Hanson [3] proved that $f(3, r) \leq 1.8r^3 + O(r^2)$. Recently, V. Rödl, L. Talysheva and I [18] proved that for every fixed k, Construction 1 by Erdős and Rado is asymptotically (in r) best possible:

Let k be fixed and r be sufficiently large. Then

$$f(k,r) = r^k + o(r^k). \tag{3}$$

We don't know how small is $o(r^k)$ in (3). I am afraid, it is the only known asymptotically exact bound concerning Δ-systems.

Abbott and B. Gardner [2] proved in 1969 that $f(3,3) = 20$, and since then no other exact value of $f(k,r)$ for $k \geq 3$ and $r \geq 3$ became known. Abbott and G. Exoo [1] obtained the lower bounds $f(k,4) \geq C \cdot 38^{k/3}$ and $f(k,6) \geq C \cdot 146^{k/3}$.

3 Weak Δ-systems

Erdős, Milner and Rado [11] gave the lower bounds $g(k,r) \geq r^k$ and $g(k,2) \geq \frac{5}{4}2^k$ for $k \geq 2$ and showed that for every positive integer k and $r > 1 + k\binom{k}{k/2}$, any k-uniform weak Δ-system is a strong Δ-system. The last result was sharpened by M. Deza [8]: he proved that for every $r > k^2 - k + 1$, any k-uniform weak Δ-system is a strong Δ-system, implying that $g(k,r) = f(k,r)$ for every $r > k^2 - k + 1$.

The lower bound on $g(k,r)$ by Erdős, Milner and Rado was obtained due to the following construction.

Construction 3. Given a $(k-1)$-uniform family $\mathcal{F}$ without weak Δ-systems of size r, a k-uniform family $\mathcal{F}'$ without weak Δ-systems of size r can be constructed from $\mathcal{F}$ by replacing every member A by the members $A_1 = A \cup \{a_1(A)\}$, $A_2 = A \cup \{a_2(A)\}, \dots, A_{r-1} = A \cup \{a_{r-1}(A)\}$, where all the elements $a_i(A)$ are distinct for all A and i. This gives

$$g(k,r) \geq (r-1)g(k-1,r) \tag{4}$$

and the bound (for $r \geq 4$) follows. The direct construction implied by this argument is as follows. Consider the complete $(r-1)$-nary tree $T_k(r)$ of height k. For every of $(r-1)^k$ pendant vertices v, let A_v be the set of the vertices of the path from v to the root w of $T_k(r)$ excluding w. The family of all these A_v is k-uniform, has $(r-1)^k$ members and contains no weak Δ-system of size r.

For $r = 3$, Erdős, Milner and Rado observed that $g(2,3) = 5$, in particular, the family of the five edges of a 5-cycle does not contain any weak Δ-system of size 3. This together with (4) gives the bound. Abbott and Hanson [4] used this observation to derive the relation $g(k,3) \geq 5g(k-2,3)$ for $k \geq 2$ and, therefore, the bound

$$g(k,3) \geq 5^{\lfloor k/2 \rfloor} 2^{k-2\lfloor k/2 \rfloor}.$$

Construction 3 is better than Construction 1 in the sense that, for given k and r, it produces the family of the same cardinality but with the stronger property. Recall that due to (3), it is asymptotically (in r) optimal for every fixed k.

The only known exact value of $g(k,r)$ for $k \geq 3$ and $r \geq 3$ is $g(3,3) = 10$ (see [4]). The best known upper bound on $g(k,3)$ due to M. Axenovich, D. G. Fon-Der-Flaass and myself [6] is:

For every $\delta > 0$, there exists a constant $C = C(\delta)$ such that

$$g(k,3) < Ck!^{0.5+\delta}.$$

Abbott and Exoo [1] gave the lower bounds $g(k,4) \geq C \cdot 10^{k/2}$ and $g(k,5) \geq C \cdot 20^{k/2}$.

4 Δ-systems in set systems with a fixed cardinality of the ground set

In [14], Erdős and Szemerédi showed

$$F(n,3) < 2^{n(1-\frac{1}{10\sqrt{n}})} \tag{5}$$

and stated that the probabilistic method implies that for each $r \geq 3$, there exists a constant $c_r > 0$, so that

$$F(n,r) > (1+c_r)^n$$

where $c_r \to 1$ as $r \to \infty$. Let

$$\beta_r = \lim_{n\to\infty} F(n,r)^{1/n}.$$

Abbott and Hanson [4] observed that β_r exists and that the probabilistic method mentioned above gives $\beta_r \geq 2(r+2)^{-1/r}$. They also presented a construction implying

$$\beta_r \geq \binom{2r-2}{r}^{1/(2r-2)} \sim \ 2^{(1-\frac{\log(2r)}{4r})}, \tag{6}$$

which is slightly better than the probabilistic bound.

The Erdős-Szemerédi proof [14] of (5) reveals relations between bounds for $f(k,r)$ and $F(n,r)$. It shows that good upper bounds for $f(k,r)$ yield satisfactory upper bounds for $F(n,r)$ and strong lower bounds (if found) for $F(n,r)$ might imply lower bounds for $f(k,r)$. W. A. Deuber, P. Erdős, D. S. Gunderson, A. G. Meyer and I [7] observed that the Erdős-Szemerédi argument together with (2) yields that for each r and sufficiently large n,

$$F(n,r) < 2^{n-\frac{\sqrt{n\log\log n}}{\log\log\log n}},$$

and that if there exists a constant C so that $f(k,3) < C^k$, then for n sufficiently large,

$$F(n,3) < 2^{n(1-0.65/C)}.$$

In particular, in this case, $\beta_3 \leq 2^{(1-1/2C)}$. It follows that if the Erdős–Rado conjecture is true, then there exists an $\epsilon > 0$ so that for large n, $F(n,3) < (2-\epsilon)^n$.

This motivates obtaining lower bounds on $F(n,r)$ and β_r. In [7], the following bound (improving (6)) is given: for every $r \geq 3$ and every n of the form $n = 2pr\lfloor \log r \rfloor$,

$$F(n,r) \geq 2^{n(1-\frac{\log\log r}{2r}-O(1/r))},$$

(and there are uniform families which witness this bound). In particular,

$$\beta_r \geq 2^{(1-\frac{\log\log r}{2r}-O(1/r))}.$$

It was also proved in [7] that for every n of the form $n = 48q+2$, $F(n,3) \geq 1.551^{n-2}$; in particular, $\beta_3 \geq 1.551$.

5 Weak Δ-systems in set systems with a fixed cardinality of the ground set

Although Construction 3 gives an exponential (in k) lower bound on $g(k,3)$, it gives only linear (in n) lower bound on $G(n,3)$. In the middle of the seventies, Abbott asked if $G(n,3)$ is superlinear in n. Answering this question, Erdős and Szemerédi [14] proved that it is superpolynomial, namely,

$$G(n,3) \geq (1+o(1))n^{\log n/4\log\log n}. \tag{7}$$

To do this, they elaborated Construction 3 as follows.

Construction 4. Take $s = \lfloor \frac{\log_2 n}{2\log_2\log_2 n} \rfloor$ disjoint copies $T_t^1, \dots, T_t^s$ of the complete binary tree T_t of height $t = \lfloor 0.5\log_2 n \rfloor$. For every $i = 2, \dots, s$, replace every vertex of T_t^i by a set of cardinality $\lfloor (\log_2 n)^{i-1} \rfloor$ (all these sets are disjoint). Let $v_1, \dots, v_s$ be some pendant vertices in $T_t^1, \dots, T_t^s$, respectively. Define $B(v_1, \dots, v_s)$ to be the union of the vertex sets of the paths connecting $v_1, \dots, v_s$ with the corresponding roots, and let $\mathcal{F}$ be the family of the sets $B(v_1, \dots, v_s)$ for all possible choices of $v_1, \dots, v_s$. Clearly,

$$|\mathcal{F}| = (2^t)^s \geq (1+o(1))2^{\frac{\log_2^2 n}{4\log_2\log_2 n}},$$

and the cardinality of the ground set is at most

$$\sum_{i=1}^{s} 2^{t+1}(\log_2 n)^{i-1} < 2^{t+1}\cdot 2\cdot (\log_2 n)^{s-1} < 2\sqrt{n}\cdot 2\cdot \frac{\sqrt{n}}{\log_2 n} < n.$$

Thus, if we prove that no three members of $\mathcal{F}$ form a weak Δ-system, then (7) follows.

Assume that members B_1, B_2 and B_3 of $\mathcal{F}$ form a weak Δ-system and that i is the largest index such that B_1, B_2 and B_3 do not coincide on T_t^i. Then, due to the structure of the binary tree, we can reorder B_1, B_2 and B_3 so that

$$|B_1 \cap B_2 \cap T_t^i| > |B_1 \cap B_3 \cap T_t^i|. \tag{8}$$

If $i = 1$, then we are done. Let $i > 1$. Since T_t^i is obtained from T_t^1 by blowing every vertex into $\lfloor (\log_2 n)^{i-1} \rfloor$ vertices, (8) yields

$$|B_1 \cap B_2 \cap T_t^i| - |B_1 \cap B_3 \cap T_t^i| \geq \lfloor (\log_2 n)^{i-1} \rfloor. \tag{9}$$

But

$$|B_3 \cap \left(\bigcup_{j=1}^{i-1} T_t^j \right) | \leq (t+1) \sum_{j=1}^{i-1} \log_2 n)^{j-1} = (1+o(1))0.5 \log_2 n \cdot (\log_2 n)^{i-2} < (\log_2 n)^{i-1}.$$

This together with (9) contradicts our assumption on B_1, B_2 and B_3.

Erdős and Szemerédi [14] also conjectured that for some $\epsilon > 0$,

$$G(n, 3) \leq (2 - \epsilon)^n.$$

This conjecture (as a consequence of a stronger result) was proved by Frankl and Rödl [15] for $\epsilon = 0.01$.

Recently, Rödl and Thoma [19] substantially improved (7) by showing that for sufficiently large n,

$$G(n, r) \geq 2^{\frac{1}{3} n^{1/5} \log_2^{4/5}(r-1)}. \tag{10}$$

To do this, they elaborated Construction 3 in a different manner than it was made in Construction 4. They replaced every vertex v in the $(r-1)$-nary tree $T_t(r)$ of height $t = \lceil 6n^{1/5} \log_2^{4/5}(r-1) \rceil$ by a set A_v of cardinality $m = \lfloor n^{3/5} \log_2^{2/5}(r-1) \rfloor$. In contrast with Construction 4, these sets A_v are not necessarily disjoint, but every two have a small intersection and the union of all A_v has the cardinality at most n. The members of the constructed family are the unions of the sets on the paths from pendant vertices of $T_t(r)$ to the root.

Later [17], this construction was elaborated to a random construction giving the bound

$$G(n, r) \geq r^{c(n \ln n)^{1/3}}.$$

Still, the gap between lower and upper bounds on $G(n, r)$ is challenging.

6 Concluding remark

One of the aims of the present article was to show that there was some progress lately in studying every of the functions $f(k, r)$, $g(k, r)$, $F(n, r)$ and $G(n, r)$, but none of the main problems is solved.

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