1. (a) Orthocenter - the intersection of the altitudes of a triangle.

(b) Isometry - a map from the plane to itself which preserves distances (i.e., $\mathbb{R}^2 \rightarrow \mathbb{R}^2$ and for all $x, y$, $d(\alpha(x), \alpha(y)) = d(x, y)$).

(c) A linear isometry - an isometry $\alpha$ with the property that $\alpha(0) = 0$.

2. (a) Any vectors $X$ and $Y$ which are perpendicular will work; for example, $X = (1, 0)$ and $Y = (0, 1)$.

(b) No such vectors can exist since the Cauchy-Schwarz inequality tells us $|X \cdot Y| \leq |X||Y|$.

(c) Rotations and central reflections.

3. See page 70 of textbook.

4. (a) Let $\alpha$ be an isometry.

Define the vector $A$ by $A = \alpha(0)$.

Define $\beta$ by $\beta = T_A \circ \alpha$.

$\beta$ is an isometry because it is the composition of two isometries.

$\beta(0) = T_A(\alpha(0)) = T_A(A) = A - A = 0$

$T_A \circ \beta = T_A \circ T_A \circ \alpha = \alpha \circ \alpha = \alpha$ \hspace{1cm} $\square$.
4. (b) \( A = \alpha(0) = -O+2C = Z(1,0) = (2,0) \)
\[
\beta(x) = I_A \circ \alpha(x) = I_A (-x+2C) = \]
\[
= I_A (-x+(2,0)) = -x+(2,0)-(2,0) = -x
\]
So \( A = (2,0), \ \beta(x) = -x \)

5. Let \( \alpha \) be an isometry. By Problem 4, there exist a vector \( A \) and a linear isometry \( \beta \) such that \( \alpha = I_A \circ \beta \).

Let \( l \) be a line. Let \( P \in l \). Then \( P = aX+bY \) where \( a+b=1 \).
Since \( \beta \) is linear,
\[
\beta(P) = \beta(aX+bY) = a\beta(X)+b\beta(Y),
\]
which is a point on \( l \beta(x) \).

So \( \beta \) maps every point on \( l \) to a point on \( l \beta(x) \).

Since \( I_A \) is a translation, it maps \( l \beta(x) \) to a line (\(* \) see proof below). So \( \alpha = I_A \circ \beta \) maps lines to lines.

\(* \) Let \( Q \in l \beta(\cdot) \). Then \( Q = c\beta(x)+d\beta(y) \) for some \( c+d = 1 \). \( I_A(Q) = c\beta(x)+d\beta(y)+A \)
\[
= c(\beta(x)+A) + d(\beta(y)+A) \text{ since } c+d=1.
\]
So \( I_A \) is a point on the line through \( \beta(x)+A \) and \( \beta(y)+A \).
5. Alternate proof.

Let \( x \) be an isometry and let \( lxy \) be a line. Let \( P \) be any point on \( lxy \) between \( x \) and \( y \). We must show that \( x(P) \) is on \( l(x)x(y) \).

Suppose \( x(P) \) is not on \( l(x)x(y) \).

Then, by the triangle inequality,
\[
d(x(x), x(y)) < d(x(x), x(P)) + d(x(P), x(y))
\]

Since \( x \) is isometry,
\[
d(x(x), x(y)) = d(x, y),
\]
\[
d(x(x), x(P)) = d(x, P), \quad d(x(P), x(y)) = d(P, y)
\]

So \( d(x, y) < d(x, P) + d(P, y) \).

However, since \( P \) is on \( lxy \) and between \( x \) and \( y \),
\[
d(x, y) = d(x, P) + d(P, y)
\]
This is a contradiction, so \( x(P) \) is on \( l(x)x(y) \). If \( P \) is on \( lxy \) but not between \( x \) and \( y \), then rename the points and proceed as above.

6. a) T by Proposition 4.10
b) T since \( |V| = \sqrt{V \circ V} \)
c) T by Corollary 3.9
d) F there are infinitely many such \( x \).