**Section 4.8** Rotations.

**Definition** The definition of a rotation agrees with our intuitive understanding. Given a center $C$ and an angle $\theta$ (could be negative), $\rho_{c,\theta}$ is rotation with center $C$ by $\theta$ degrees counterclockwise.

$\rho_{c,\theta}(C) = C$ and for $X \neq C$,

$X' = \rho_{c,\theta}(X)$ is a point such that $d(C, X') = d(C, X)$ and $\angle (X-C, X'-C) = \theta$

![Diagram of rotation](attachment:image.png)

**Theorem 4.27** A rotation is an isometry.

**Proof** We'll accept this theorem without proof as it is intuitively clear that rotating preserves all distances.

**Theorem 4.28** Given a rotation $\rho_{c,\theta}$, let $m$ and $n$ be any two lines through $C$ so that $\angle(m, n) = \frac{\theta}{2}$.

Then $\rho_{c,\theta}^{-1} = \sigma_n \sigma_m$
Proof of Theorem 4.28

We are given a fixed line $m$ and two points $C$ with $d(m, C) = d(C, N)$.

Let $M$ be on $m$, $N$ on $m$, so that $d(C, M) = d(C, N)$.

Let $N' = \sigma_m(N)$ and $M' = \sigma_n(M)$.

We will show that $\rho_{C, 0}$ and $\sigma_n \sigma_m$ agree on the three non-collinear points $C, M, N'$, which by Exercise 4.2 will imply $\rho_{C, 0} = \sigma_n \sigma_m$.

$C$ is fixed by both $\rho_{C, 0}$ and $\sigma_n \sigma_m$.

$d(C, M) = d(C, M')$ and $\sigma_n \sigma_m$ preserve angles,

$$\left( M-C, M'-C \right) = \left( M-C, N-C \right) + \left( N-C, M'-C \right) = 90^\circ + 90^\circ = 180^\circ$$

so $\rho_{C, 0} (M) = M'$.

also $\sigma_n \sigma_m (M) = \sigma_n (M) = M'$.

By argument similar to above, $\rho_{C, 0} (N') = N$ and $\sigma_n \sigma_m (N') = \sigma_n (N) = N$.

$\therefore \rho_{C, 0} = \sigma_n \sigma_m$. \qed
Theorem 4.30. An isometry with exactly one fixed point \( C \) is a rotation with center \( C \).

**Proof:** Theorem 4.28 and Theorem 4.18 \( \blacksquare \).

Theorem 4.29. (This will be needed in the classification theorem for isometries.)

Let \( m, n, p \) be lines which all intersect at a point \( C \). Then there is a line \( g \) through \( C \), so that

\[
T_n \circ T_m \circ T_p = T_g.
\]

**Proof.** \( T_n \circ T_m \) is a rotation \( p, \theta \), where \( \theta = \alpha L(m, n) \). Let \( g \) be a line through \( C \) such that \( L(p, g) = \theta \).

Then, by Theorem 4.28, \( p, \theta = T_g \circ T_p \).

Since \( p, \theta \) equals both \( T_n \circ T_m \) and \( T_g \circ T_p \),

\[
T_n \circ T_m = T_g \circ T_p \ 	ext{so}
\]

\[
T_n \circ T_m \circ T_p = T_g \circ T_p \circ T_p = T_g \circ I = T_g \ 	ext{\( \blacksquare \).}
\]