Homework #6 Solutions

3.4/1

\[ k = \text{spring constant} = 16 \quad m = 4 \]

\[ F_s = -kx = -16x \quad \text{This is the only force.} \]

\[ -16x = 4x'' \quad 4r^2 + 16 = 0 \]

\[ r = \pm 2i \quad x = c_1 \cos 2t + c_2 \sin 2t \]

Since \( \cos u \) has period \( 2\pi \), \( \cos 2t \) has period \( \pi \). Same for \( \sin 2t \).

Period: \( \pi \) sec./cycle

Frequency: \( \frac{1}{\pi} \) cycles/sec (Hz), which is \( 2 \) radians/sec.

3.4/3

To find spring constant, \( 15 = k \cdot 0.2 \text{ cm} \).

\[ 15 = k \cdot m \quad k = 75 \]

\[ m = 3 \]

\[ F_s = -kx = -75x \]

\[ mx'' = F_s \]

\[ 3x'' = -75x \]

\[ x'' + 25x = 0 \]

\[ r^2 + 25 = 0 \]

\[ r = \pm 5i \]

\[ x = c_1 \cos 5t + c_2 \sin 5t \]

cont'd
initial conditions \( x(0) = 0 \), \( x'(0) = -10 \) m/s
\( 0 = x(0) = c_1 \),
\( x' = -5c_1 \sin 5t + 5c_2 \cos 5t \)
\(-10 = x'(0) = 5c_2 \) so \( c_2 = -2 \)
Solution is \( x(t) = -2 \sin 5t \)

Amplitude : 2
Period : \( \frac{2\pi}{5} \) sec/cycle
Frequency : \( \frac{5}{2\pi} \) Hz or \( 5 \) radian/sec.

To find spring constant \( g = k \left( \frac{4}{m} \right) \)
\( k = 36 \)
\( m = 25 \Delta g = 0.25 \) kg
\( 0.25x'' = -36x \)
\( 0.25x'' + 36x = 0 \)
\( x'' + 144x = 0 \)
\( r^2 + 144 = 0 \)
\( r = \pm 12i \)

Solution
initial conditions : \( x(0) = 1 \) m. \( x'(0) = -5 \) m/s
\( x' = -12c_1 \sin(12t) + 12c_2 \cos(12t) \)
\( 1 = x(0) = c_1 \)
\(-5 = x'(0) = 12c_2 \) so \( c_2 = -\frac{5}{12} \)

Solution : \( x(t) = c_1 \cos(12t) - \frac{5}{12} \sin(12t) \)
3.4/4 cont'd

\[ C = \sqrt{1^2 + (\frac{5}{12})^2} = \sqrt{\frac{169}{144}} = \frac{13}{12} \]

\[ \cos \alpha = \frac{1}{\frac{13}{12}} = \frac{12}{13} \quad \sin \alpha = \frac{\frac{-5}{12}}{\frac{13}{12}} = -\frac{5}{13} \]

Then \[ x(t) = \frac{13}{12} \left( \frac{12}{13} \cos(12t) - \frac{5}{13} \sin(12t) \right) \]

\[ = \frac{13}{12} \left( \cos \alpha \cos(12t) + \sin \alpha \sin(12t) \right) \]

\[ x(t) = \frac{13}{12} \cos \left( 12t - \alpha \right) \]

where \( \cos \alpha = \frac{12}{13}, \quad \sin \alpha = -\frac{5}{13} \)

Using a calculator,

\[ \sin^{-1} \left( -\frac{5}{13} \right) \approx -0.3948 \]

From diagram at left,

this number is \( \alpha \approx 2\pi \),

so \( \alpha = 2\pi - 0.3948 \approx 5.8884 \)

Amplitude: \( \frac{13}{12} \) m.

Period: \( \frac{2\pi}{12} = \frac{\pi}{6} \) sec (per cycle)
\[ m = 0.75 \quad k = 48 \]
\[ 3x'' + 48x = 0 \]
\[ \frac{4}{3}(0.75x'' + 48x = 0) \]
\[ x'' + 64x = 0 \]
\[ r^2 + 64 = 0 \quad r = \pm 8i \]
\[ x(t) = c_1 \cos(8t) + c_2 \sin(8t) \]

- **Period:** \( \frac{2\pi}{8} = \frac{\pi}{4} \text{ sec/cycle} \)
- **Frequency:** \( \frac{1}{4} \text{ cycles/sec} = 8 \text{ rad/sec} \)

\[ m = 3 \quad c = 30 \quad k = 63 \quad x_0 = 2 \quad v_0 = 2 \]
\[ 3x'' + 30x' + 63x = 0 \]
\[ \frac{1}{3}(3r^2 + 30r + 63 = 0) \]
\[ r^2 + 10r + 21 = 0 \]
\[ (r + 7)(r + 3) = 0 \quad r = -7, -3 \]
\[ x(t) = c_1 e^{-7t} + c_2 e^{-3t} \]
\[ x'(t) = -7c_1 e^{-7t} - 3c_2 e^{-3t} \]

1. \( a = x(0) = c_1 + c_2 \)
2. \( a = x'(0) = -7c_1 - 3c_2 \)

From (1), \( c_2 = 2 - c_1 \)

Plug into (2):

\[ a = -7c_1 - 3(2 - c_1) = -7c_1 - 6 + 3c_1 = -4c_1 - 6 \]
\[ 8 = -4c_1 \quad \boxed{c_1 = -2} \]

Then \( c_2 = 2 - c_1 = 2 - (-2) = 4 \)

**Position function:** \( x(t) = -2e^{-7t} + 4e^{-3t} \)

Motion is **overdamped** (2 distinct real roots \( r \))

Find undamped position function:

\[ 3x'' + 63x = 0 \quad x_0 = 2, \quad v_0 = 2 \]

\[ \text{cont'd} \]
\[ 3 \pi^2 + 63 = 0 \]
\[ \pi^2 + 21 = 0 \quad \pi = \pm \sqrt{21} i \]

\[ x(t) = c_1 \cos(\sqrt{21} t) + c_2 \sin(\sqrt{21} t) \]
\[ x'(t) = -\sqrt{21} c_1 \sin(\sqrt{21} t) + \sqrt{21} c_2 \cos(\sqrt{21} t) \]
\[ 2 = x(0) = c_1 \]
\[ 2 = x'(0) = \sqrt{21} c_2 \quad \text{so} \quad c_2 = \frac{2}{\sqrt{21}} \]

**Undamped:**
\[ x(t) = 2 \cos(\sqrt{21} t) + \frac{2}{\sqrt{21}} \sin(\sqrt{21} t) \]
\[ C = \sqrt{4 + \frac{4}{21}} = \sqrt{\frac{88}{21}} = 2 \sqrt{\frac{22}{21}} \]

**Rewrite:**
\[ x(t) = 2 \sqrt{\frac{22}{21}} \left[ \frac{\sqrt{22}}{\sqrt{21}} \cos(\sqrt{21} t) + \frac{1}{\sqrt{22}} \sin(\sqrt{21} t) \right] \]

\[ x(t) = 2 \sqrt{\frac{22}{21}} \cos(\sqrt{21} t - \alpha) \quad \text{call this } u(t) \]

Where \( \cos \alpha = \frac{\sqrt{22}}{\sqrt{21}} \quad \sin \alpha = \frac{1}{\sqrt{22}} \).

Using calculator, \( \alpha \approx 0.2 \) radians
m = 1  c = 8  k = 16  x_0 = 5  v_0 = -10

\[ x'' + 8x' + 16x = 0 \]
\[ r^2 + 8r + 16 = 0 \]
\[ (r + 4)^2 = 0, \quad r = -4, -4 \]

\[ x(t) = c_1 e^{-4t} + c_2 te^{-4t} \]
\[ 5 = x(0) = c_1 \]
\[ x'(t) = -4c_1 e^{-4t} + c_2 e^{-4t} - 4c_2 t e^{-4t} \]
\[ -10 = x'(0) = -4c_1 + c_2 = -20 + c_2 \quad \text{so} \quad c_2 = 10 \]

**Position function:**
\[ x(t) = 5e^{-4t} + 10te^{-4t} \]

Critically damped because repeated real root r

**Undamped motion:**
\[ u'' + 16u = 0 \quad u_0 = 5, \quad v_0 = -10 \]
\[ r^2 + 16 = 0 \quad r = \pm 4i \]

\[ u(t) = c_1 \cos 4t + c_2 \sin 4t \]
\[ u'(t) = -4c_1 \sin 4t + 4c_2 \cos 4t \]
\[ 5 = u(0) = c_1 \]
\[ -10 = u'(0) = 4c_2 \quad c_2 = -\frac{5}{2} \]

**Undamped position fn:**
\[ u(t) = 5 \cos 4t - \frac{5}{2} \sin 4t \]

\[ C = \sqrt{5^2 + (\frac{5}{2})^2} = \sqrt{\frac{25}{4}} = \frac{5}{2} \sqrt{5} \]

\[ u(t) = \frac{5\sqrt{5}}{2} \left( \frac{2}{\sqrt{5}} \cos 4t - \frac{1}{\sqrt{5}} \sin 4t \right) \]

\[ u(t) = \frac{5\sqrt{5}}{2} \cos (4t - \alpha) \quad \text{where} \quad \cos \alpha = \frac{\sqrt{5}}{2}, \quad \sin \alpha = -\frac{1}{\sqrt{5}} \]

Using calculator, \( \alpha \approx -0.5 \)

\[ \frac{5\sqrt{5}}{2} \approx 5.6 \]
\[ \begin{align*}
\text{3.4/17 cont'd} \\
\text{3.4/18} \quad m=2, \quad c=12, \quad k=50 \quad x_0=0, \quad v_0=-8 \\
2x'' + 12x' + 50x = 0 \\
2r^2 + 12r + 50 = 0 \\
r^2 + 6r + 25 = 0 \\
r = \frac{-6 \pm \sqrt{36-400}}{2} = \frac{-6 \pm \sqrt{-64}}{2} = -3 \pm 4i \\
x(t) = e^{-3t}(c_1 \cos 4t + c_2 \sin 4t) \\
x'(t) = -3e^{-3t}(c_1 \cos 4t + c_2 \sin 4t) + e^{-3t}(-4c_1 \sin 4t + 4c_2 \cos 4t) \\
0 = x(0) = c_1 \\
-8 = x'(0) = -3(0+c_1) + 1(0+4c_2) \quad \text{so} \quad c_2 = -2 \\
x(t) = -2e^{-3t} \sin 4t \quad \text{Position function} \\
\text{Underdamped since complex roots r} \\
\text{cont'd}
\end{align*}\]
Note: This seems to be a perfectly good form for \( x(t) \), but if we really want it written in terms of \( \cos \), it would be

\[
x(t) = -2e^{-3t} \cos \left( \frac{\pi}{2} - \frac{\pi}{2} \right)
\]

**Undamped motion:**

\[
2u'' + 50u = 0 \quad u_0 = 0 \quad v_0 = -8
\]

\[
2r^2 + 50 = 0
\]

\[
r^2 + 25 = 0
\]

\[
r = \pm 5i
\]

\[
u(t) = c_1 \cos 5t + c_2 \sin 5t
\]

\[
u'(t) = -5c_1 \sin 5t + 5c_2 \cos 5t
\]

\[
0 = u(0) = c_1
\]

\[
-8 = u'(0) = 5c_2 \quad \text{so} \quad c_2 = -\frac{8}{5}
\]

\[
u(t) = -\frac{8}{5} \sin (5t)
\]

As with \( x \), this seems like a good form. We can also write

\[
u(t) = -\frac{8}{5} \cos (5t - \frac{\pi}{2})
\]
IODE Project III  (partial solutions)

1.a. \( x'' + 15x' + .5x = 0 \)
\( 2(r^2 + 15r + .5) = 0 \)
\( 2r^2 + 30r + 1 = 0 \)
\( (2r + 1)(r + 1) = 0 \)
\( r = -\frac{1}{2}, -1 \)
\( x(t) = c_1 e^{-\frac{1}{2}t} + c_2 e^{-t} \)

c. 100% of solutions decay. See \( x(t) \) from part (a). No matter what \( c_1 \) and \( c_2 \) are, \( x(t) \to 0 \) as \( t \to \infty \)

2.a. \( x'' - \sqrt{3}x' - .25x = 0 \)
\( 4r^2 - 4\sqrt{3}r - 1 = 0 \)
\( r = \frac{\sqrt{3} \pm \sqrt{3 + 4}}{2} \)
\( r = \frac{\sqrt{3} \pm \sqrt{3 + 1}}{2} \)
\( r = \frac{\sqrt{3} \pm 2}{2} = \frac{\sqrt{3} \pm 1}{2} \)
\( x(t) = c_1 e^{(\sqrt{3} - 1)t} + c_2 e^{(\sqrt{3} + 1)t} \)

c. Note \( \frac{\sqrt{3} - 1}{2} < 0 \) and \( \frac{\sqrt{3} + 1}{2} > 0 \)

If \( c_2 = 0 \), then \( x(t) \) decays.
If \( c_2 \neq 0 \), then \( x(t) \) grows, since the first term decays and the second grows, so their sum grows.
IODE III

2c. cont'd

Therefore 100% of the solutions grow (which seems strange, since some of them decay, but remember we have decay only for one $c_2$ value ($c_2=0$) out of the infinite number of $c_2$ values, and 1 out of infinity is 0% decay, so 100% growing).

$$3a. \quad x'' - 0.5x' + 5x = 0$$
$$r^2 - 0.5r + 5 = 0$$
$$2r^2 - r + 1 = 0$$

$$r = \frac{1 \pm \sqrt{1-8}}{2} = \frac{1 \pm \sqrt{-7}}{2} = \frac{1 \pm i\sqrt{7}}{2}$$

$$x(t) = e^{\frac{-1}{2}t} (c_1 \cos(\frac{\sqrt{7}}{2}t) + c_2 \sin(\frac{\sqrt{7}}{2}t))$$

c. Grows while oscillating 100% of the time. There is only one solution for which this does not happen, which is when $c_1 = c_2 = 0$ so $x(t) \equiv 0$. 
4a.

\[ x^{(4)} + 16 x'' + 100 x = 0 \]
\[ r^4 + 16 r^2 + 100 = 0 \]
\[ (r^2 + 2r + 10)(r^2 - 2r + 10) = 0 \]
\[ r = \frac{-2 \pm \sqrt{4 - 40}}{2} = \frac{-2 \pm \sqrt{-36}}{2} = -1 \pm 3i \]

or

\[ r = \frac{2 \pm \sqrt{4 - 40}}{2} = 1 \pm 3i \]

\[ x(t) = e^{-t} (c_3 \cos 3t + c_2 \sin 3t) + e^{-t} (c_3 \cos 3t + c_4 \sin 3t) \]

b. 100% of solutions will grow while oscillating. This is because out of the infinite number of choices for \( c_3, c_4 \), we always get growth while oscillating except for the one choice \( c_3 = c_4 = 0 \) (then we get decay while oscillating). One out of infinity is 0%.

5. a. decays
b. grows
c. grows
d. decays while oscillating
e. oscillates
f. grows while oscillating