0. \( y'' - y = 0 \).

a. Verify \( y_1 = e^x \) is a solution: \( y_1'' = e^x \), so
   \( y_1'' - y_1 = 0 \). \( y_1 \) satisfies the given D.E.

b. Verify \( y_2 = e^{-x} \) is a solution:
   \( y_2' = -e^{-x} \)
   and \( y_2'' = e^{-x} \), so \( y_2'' - y_2 = 0 \).

c. Let \( y = c_1 e^x + c_2 e^{-x} \)
   then \( y' = c_1 e^x - c_2 e^{-x} \)

Plug in \( y(0) = 0 \), \( y'(0) = 5 \)
   \( 0 = c_1 + c_2 \); \( 5 = c_1 - c_2 \).
We get \( c_1 = \frac{5}{2} \) and \( c_2 = -\frac{5}{2} \).
The particular solution is
   \( y = \frac{5}{2} e^x - \frac{5}{2} e^{-x} \).

3. \( y'' + 4y = 0 \).

a. Verify \( y_1 = \cos 2x \) is a solution:
   \( y_1' = -2\sin 2x \) and \( y_1'' = -4\cos 2x \) so \( y_1'' + 4y_1 = 0 \).

b. Verify \( y_2 = \sin 2x \) is a solution:
   \( y_2' = 2\cos 2x \) and \( y_2'' = -4\sin 2x \) so \( y_2'' + 4y_2 = 0 \).
c. Let \( y = c \cos 2x + C_2 \sin 2x \)
\[
y' = -2c\sin 2x + 2C_2 \cos 2x
\]
Plug in \( y(0) = 3 \) and \( y'(0) = 8 \)
\[
3 = c \cdot 0 + C_2 \cdot 0 ; \quad 8 = 0 + 2C_2 \quad \text{so} \quad C_2 = 4 \quad \text{and} \quad c = 3
\]
The particular solution is
\[
y = 3 \cos 2x + 4 \sin 2x
\]

10. \( y'' - 10y' + 25y = 0 \)

a. Verify \( y_1 = e^{5x} \) is a solution:
\[
y_1' = 5e^{5x}, \quad y_1'' = 25e^{5x} \quad \text{so} \quad y_1'' - 10y_1' + 25y_1 = 25e^{5x} - 50e^{5x} + 25e^{5x} = 0.
\]

b. Verify \( y_2 = xe^{5x} \) is a solution:
\[
y_2' = e^{5x} + 5xe^{5x}, \quad y_2'' = 10e^{5x} + 25xe^{5x}
\]
\[
y_2'' - 10y_2' + 25y_2 = 10e^{5x} + 25xe^{5x} - 10(e^{5x} + 5xe^{5x}) + 25xe^{5x} = 0
\]
so \( y_2 \) satisfies \( y'' - 10y' + 25y = 0 \).
Section 3.1

10) cont’d
6) Let \( y = ce^{5x} + 6xe^{5x} \)
then \( y' = 5ce^{5x} + 6e^{5x} + 56xe^{5x} \)
Plug in \( y(0) = 3 \) and \( y'(0) = 13 \)

\[ 3 = c + 0 \quad 13 = 5c + c_2 + 0 \]
so \( c_1 = 3 \) and \( c_2 = -2 \)
The particular solution is
\[ y = 3e^{5x} - 2xe^{5x} \]

17) Let \( y = \frac{1}{x} \). Then \( y' = -\frac{1}{x^2} \). \( y' + y^2 = \frac{-1}{x} + \frac{1}{x^2} = 0 \), so \( y = \frac{1}{x} \) is a solution of \( y' + y^2 = 0 \).

Now consider \( y = \frac{c}{x} \). \( y' = -\frac{c}{x^2} \).
Then \( y' + y^2 = -\frac{c}{x^2} + \frac{c^2}{x^2} = \frac{c^2 - c}{x^2} \).
This is not \( \leq 0 \) (unless \( c = 0 \) or 1), so \( y = \frac{c}{x} \) is not a solution of \( y' + y^2 = 0 \).
(Notice \( y' + y^2 = 0 \) is not linear, so the Superposition Theorem does not apply).
18. Consider \( y = x^3 \).

\[ y' = 3x^2 \quad \text{and} \quad y'' = 6x. \]

\[ yy'' = x^3(6x) = 6x^4, \quad \text{so} \quad y = x^3 \quad \text{is a solution of} \quad yy'' = 6x^4. \]

Now consider \( y = cx^3 \). \( y' = 3cx^2 \), \( y'' = 6cx \).

Then \( yy'' = (cx^3)(6cx) = 6c^2x^4 \).

So \( y = cx^3 \) is not a solution of \( yy'' = 6x^4 \) (unless \( c = 1 \) or \(-1\)).

Since \( yy'' = 6x^4 \) is not linear, this does not violate the Superposition Theorem.

19. Let \( y_1 = 1 \). Then \( y' = 0 \) and \( y'' = 0 \).

So \( yy'' + (y')^2 = 1 \cdot 0 + 0^2 = 0 \) so \( y_1 \) is a solution of \( yy'' + (y')^2 = 0 \).

Let \( y_2 = \sqrt{x} \). Then \( y_2' = \frac{x}{2} \) and \( y_2'' = \frac{1}{4} x^{-3/2} \)

so \( y_2 y'' + (y')^2 = x^{3/2}(\frac{1}{4} x^{-3/2}) + (\frac{x}{2})^2 = \frac{1}{4} x^1 + \frac{1}{4} x^1 = 0 \).

Therefore \( y_2 = \sqrt{x} \) is also a solution of \( yy'' + (y')^2 = 0 \).
Section 3.1

19 cont'd.

Now let \( y_3 = 1 + \sqrt{x} \).

\[ y_3' = \frac{1}{2} x^{-\frac{1}{2}} \] and \[ y_3'' = \frac{1}{2^2} x^{-\frac{3}{2}} \]

Then \( y_0 y_3'' + (y_3')^2 = (1 + x^\frac{1}{2}) \left( \frac{1}{2^2} x^{-\frac{3}{2}} \right) \left( \frac{1}{2} x^{-\frac{1}{2}} \right)^2 \)

\[ = -\frac{1}{4} x^{-\frac{3}{2}}. \]

So \( y_3 = y_1 + y_2 \) is not a solution of \( y y'' + (y')^2 \).

Since \( y y'' + (y')^2 \) is not linear, this does not contradict the Superposition Theorem.

24 \( f(x) = \sin^2 x; \ g(x) = 1 - \cos 2x \) are linearly dependent on the real line because \( f(x) = 2g(x) \), so \( f \) is a constant multiple of \( g \).

25 \( f(x) = e^x \sin x \) and \( g(x) = e^x \cos x \) are linearly independent on the real line because neither is a constant multiple of the other.
We know $y_p$ is a solution of 
\[ y'' + py' + qy = f(x), \] so 
\[ y_p'' + p(x)y_p' + q(x)y_p = f(x). \]

We also know $y_c$ is a solution of 
\[ y'' + p(x)y' + q(x)y = 0, \] so 
\[ y_c'' + p(x)y_c' + q(x)y_c = 0. \]

Now put $y_c + y_p$ into the left hand side of the equation
\[
(y_c + y_p)'' + p(x)(y_c + y_p)' + q(x)(y_c + y_p)
\]
\[= (y_c'' + p(x)y_c' + q(x)y_c) + (y_p'' + p(x)y_p' + q(x)y_p)\]

Using (1) and (2), this = $0 + f(x) = f(x)$.

Therefore $y = y_c + y_p$ is a solution of 
\[ y'' + py' + qy = f(x). \]
Consider \( y_1 = x^2 \)

\[ y_1' = 2x \quad \text{and} \quad y_1'' = 2, \quad \text{so} \]

\[ x^2 y_1'' - 4x y_1' + 6y_1 = x^2(2) - 4x(2x) + 6x^2 = 0 \]

Also \( y_1(0) = 0 \) and \( y_1'(0) = 0 \).

Consider \( y_2 = x^3 \)

\[ y_2' = 3x^2 \quad \text{and} \quad y_2'' = 6x. \]

So

\[ x^2 y_2'' - 4x y_2' + 6y_2 = x^2(6x) - 4x(3x^2) + 6x^3 = 0 \]

Also \( y_2(0) = 0 \) and \( y_2'(0) = 0 \).

Both \( y_1 \) and \( y_2 \) are solutions of the initial value problem \( x^2 y'' - 4x y' + 6y = 0, \ y(0)=y'(0)=0 \).

This does not contradict Theorem 2 because for theorem 2, the O.E. must be in the form \( y'' + p(x)y' + q(x)y = 0 \) with \( p, q, F \) continuous on an open interval containing \( a = 0 \).

If we put our equation in this form by dividing by \( x^2 \), \( (y'' - \frac{4}{x} y' + \frac{6}{x^2} = 0) \), we see that \( p(x) \) and \( q(x) \) are not continuous at 0.