Math 385, Section D1 - Test #3 - December 9, 2005

Time: 55 minutes. You may not use any books or notes or calculator. There are 100 points possible. To get full credit, you must show your work.

The following formulas are given:

\[ x_{sp}(t) = \sum_{n=1}^{\infty} \frac{B_n \sin(\omega_n t - \alpha_n)}{\sqrt{(k - m\omega_n^2)^2 + (\omega_n)^2}} \]

\[ u(x, t) = \sum_{n=1}^{\infty} a_n \exp(-n^2 \pi^2 kt/L^2) \sin \frac{n\pi x}{L} \]

\[ u(x, t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \exp(-n^2 \pi^2 kt/L^2) \cos \frac{n\pi x}{L} \]
1. (a) (5 points) Give the definition of even function.

\[ f \text{ is even if } f(-t) = f(t) \text{ for all } t \text{ in the domain of } f. \]

(b) (10 points) Suppose that \( f \) is an even function. Show that

\[ \int_{-a}^{0} f(t) \, dt = \int_{0}^{a} f(t) \, dt. \]

Substitution: \( u = -t \quad dt = -du \) when \( t = -a, \)

\[ \int_{-a}^{0} f(t) \, dt = \int_{a}^{0} f(-u) \, du = \int_{0}^{a} f(-u) \, du \]

\[ = \int_{0}^{a} f(u) \, du \quad \text{(since } f \text{ is even)} \]

\[ = \int_{0}^{a} f(t) \, dt \]
2. Let \( f(t) \) be a function of period 4, given by \( f(t) = t \) for \( 0 < t < 4 \).

(a) (6 points) Sketch the graph of \( f(t) \), including at least three full periods.

(b) (10 points) Write the Fourier series for \( f(t) \). Important note: you may give the Fourier coefficients \( a_n \) and \( b_n \) as definite integrals, without evaluating those integrals.

\[
\frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos \frac{n\pi t}{2} + b_n \sin \frac{n\pi t}{2}
\]

where \( a_0 = \frac{1}{2} \int_{0}^{4} t \cos \frac{n\pi t}{2} \, dt \)

\( a_n = \frac{1}{2} \int_{0}^{4} t \cos \frac{n\pi t}{2} \, dt \)

\( b_n = \frac{1}{2} \int_{0}^{4} t \sin \frac{n\pi t}{2} \, dt \)
3. (15 points) Find the formal Fourier series solution of the endpoint value problem

\[ x'' - 4x = 1, \quad x(0) = x(2) = 0. \]

**Extend \( f \) as odd, period 4**

\[ f(t) \quad L = 2 \]

**Fourier coefficients:** \( a_n = 0 \) since \( f \) is even, odd

\[
\begin{align*}
    b_n &= \frac{1}{2} \int_{-2}^{2} f(t) \sin \frac{n\pi t}{2} \, dt = 2 \cdot \frac{1}{2} \int_{0}^{2} 1 \cdot \sin \frac{n\pi t}{2} \, dt \\
    &= -\frac{2}{n\pi} \cos \frac{n\pi t}{2} \bigg|_{0}^{2} = -\frac{2}{n\pi} \left( \cos n\pi - 1 \right) = \frac{4}{n\pi} \quad \text{if} \quad n \text{ odd} \\
    &= 0 \quad \text{if} \quad n \text{ even}
\end{align*}
\]

\[ X = \sum_{n \text{ odd}} b_n \sin \frac{n\pi t}{2} \]

\[ x'' = \sum_{n \text{ odd}} -\frac{n^2\pi^2}{4} b_n \sin \frac{n\pi t}{2} \]

\[ x'' - 4x = \sum_{n \text{ odd}} b_n \left( -\frac{n^2\pi^2}{4} - 4 \right) \sin \frac{n\pi t}{2} = f(t) = \sum_{n \text{ odd}} \frac{4}{n\pi} \sin \frac{n\pi t}{2} \]

\[ \beta_n = \frac{4}{n\pi} / \left( -\frac{n^2\pi^2}{4} - 4 \right) \]

and \( x(t) = \sum_{n \text{ odd}} \beta_n \sin \frac{n\pi t}{2} \)
4. A rod 5 cm long is made of a material with "thermal diffusivity constant" $k = 0.1 \text{ cm}^2/\text{s}$ and has insulated lateral surfaces. It is heated to a temperature of $f(x) = \sin \frac{\pi x}{5}$ degrees and at time $t = 0$, its two ends are embedded in ice at 0 degrees.

(a) (10 points) Find the temperature $u(x,t)$ of the rod. Important note: You can find the Fourier series for the function $f(x)$ without computing any integrals - look at the form of $f(x)$.

$$u(x,t) = \sum_{n=0}^{\infty} e^{-\frac{\pi^2 n^2 t}{25}} \sin \frac{\pi x}{5}$$

(b) (6 points) What differential equation does $u(x,t)$ satisfy? In other words, give the heat equation.

$$u_t = 0.1 \ u_{xx}$$

(c) (6 points) What is the temperature of the midpoint of the rod after 10 minutes (600 seconds)?

$$x = \frac{5}{2}, \ t = 600 \Rightarrow u\left(\frac{5}{2}, 600\right) = e^{-\frac{\pi^2 600}{25}} \sin \frac{\pi}{2} = e^{-\frac{\pi^2 600}{25}}$$
5. (12 points) Consider a mass-and-spring system with mass \( m = 3 \) and Hooke's constant \( k = 12 \). Suppose the system is under the influence of a periodic forcing function \( F(t) \) which has Fourier series

\[
F(t) = \sum_{n=1}^{\infty} \frac{3\pi}{n} \sin n\pi t.
\]

Does pure resonance occur? Explain your answer.

\[
3x'' + 12x = 0 \quad \text{(unforced)}
\]
\[
x'' + 4x = 0
\]

natural frequency = 2.

Can \( n\Pi = 2 \)? \( \text{No, since } \frac{2}{n} \text{ is integer} \)

2 is not one of the frequencies of \( F(t) \), so pure resonance does not occur.
6. (4 points each part) Answer True or False for each part. No explanation is needed and this question has no partial credit, just right or wrong.

(a) For any piecewise smooth function \( f(t) \) which is defined for all \( t \), the Fourier series is defined and converges.

\[
F \quad \text{(must be periodic)}
\]

(b) Let \( f(t) \) be the odd function of period 2 which is defined as \( f(t) = 4 \) for \( 0 < t < 1 \). When \( t = 1 \), the Fourier series converges to 0.

\[
T \quad \text{converges to} \quad \frac{1}{2}(f(1^+) + f(1^-)) = \frac{1}{2}(4 - 4) = 0
\]

(c) If \( f(t) \) is periodic with period 10, then \( \int_{0}^{10} f(t) \, dt = \int_{-5}^{5} f(t) \, dt \).

\[
T \quad \int_{0}^{10} f(t) \, dt = \int_{a}^{a+10} f(t) \, dt \quad \text{for any} \ a.
\]

(d) The function \( f(x) \) which is defined as \( f(x) = 1/x \) for \( x > 0 \) and \( f(x) = 2 \) for \( x \leq 0 \) is piecewise continuous.

\[
F \quad \text{false} \quad \text{the discontinuity at} \ x = 0 \ \text{is not a jump discontinuity.}
\]

(e) The method called "separation of variables" is used to find the steady periodic solution of an equation of the form \( m \ddot{x} + cx' + kx = F(t) \), where \( F(t) \) is periodic.

\[
F \quad \text{It is used for the heat equation.}
\]