Homework #3  Solutions to some problems
242  Fall '05

11.4/4  \( x(t) = 17 - 17t; \ y(t) = -13 + 13t; \ z(t) = -31 + 31t \)

11.4/6  direction vector \( \overrightarrow{P_2 P_1} = (3, -13, 3) \)

Using \( P_1 \) as the point,
\[
\begin{align*}
  x(t) &= 3 + 3t; \quad y(t) = 5 - 13t; \quad z(t) = 7 + 3t \\
\end{align*}
\]

Note: if you use \( P_2 \) instead of \( P_1 \), you’ll get different parametric equations which are also correct.

11.4/10 direction vector \( \overrightarrow{PQ} = (2, -2, 15) \)

Using \( P_1 \), parametric eqs.
\[
\begin{align*}
  x(t) &= 2 + 2t; \quad y(t) = 5 - 2t; \quad z(t) = -7 + 15t \\
\end{align*}
\]

Solve for \( t \):
\[
\begin{align*}
  t &= \frac{x-2}{2} \quad t = \frac{y-5}{-2} \quad t = \frac{z+7}{15} \\
  \frac{x-2}{2} = \frac{y-5}{-2} = \frac{z+7}{15} \quad \text{symmetric eqs.}
\end{align*}
\]
The direction vector for $L_1$ is $\langle 4, 1, -2 \rangle$
The direction vector for $L_2$ is $\langle 6, -3, 8 \rangle$
These are not scalar multiples of one another, so $L_1$ and $L_2$ are not parallel.

Parametric Eqs. for $L_1$ are

$\begin{align*}
&x = 11 + 4t; \quad y = 6 + t; \quad z = -5 - 2t \\
\end{align*}$

And for $L_2$

$\begin{align*}
&x = 13 + 6s; \quad y = 2 - 3s; \quad z = 5 + 8s \\
\end{align*}$

If $L_1$, $L_2$ intersect, then

$\begin{align*}
\begin{cases}
11 + 4t = 13 + 6s & (1) \\
6 + t = 2 - 3s & (2) \\
-5 - 2t = 5 + 8s & (3)
\end{cases}
\end{align*}$

Solve (2) for $t$ and plug into (3) to get

$-5 - 2(-4 - 3s) = 5 + 8s$

Then from (2),

$\begin{align*}
&3 - 3s = 5 + 8s \\
&-2 = 11s \\
&s = -\frac{2}{11}
\end{align*}$

Put this $s,t$ into (1):

$11 + 4\left(-\frac{38}{11}\right) = 13 + 6\left(-\frac{2}{11}\right)$

This is false

(work it out).

There is no point of intersection; skew
11.4/20  The direction vector for \( L_1 \) is \( \langle 12, 20, -28 \rangle \)
And for \( L_2 \) \( \langle 9, 15, -21 \rangle \)
These are parallel because \( \langle 12, 20, -28 \rangle = \frac{4}{3} \langle 9, 15, -21 \rangle \)
\( L_1 \) and \( L_2 \) are \( \boxed{\text{parallel}} \).

11.4/22  \( \vec{n} \cdot (\vec{r} - \langle 3, 4, 5 \rangle) = 0 \)
Let \( \vec{r} = \langle x, y, z \rangle \)

\[ -2(x-3) + 7(y+4) + 3(z-5) = 0. \]

11.4/30  The normal vector to \( x+y-2z=0 \)
is \( \vec{n} = \langle 1, 1, -2 \rangle \).
Answer: \( (x-5) + (y-1) - 2(z-4) = 0 \)

11.4/37  For \( L \), the direction vector is \( \langle 2, -5, 3 \rangle \)
For \( \partial \), the normal vector is \( \vec{n} = \langle 3, 3, -4 \rangle \)
Since \( \vec{n} \cdot \vec{v} \neq 0 \), \( \vec{v} \) and \( \vec{n} \) are not perpendicular, so \( L \) is not parallel to \( \partial \).

cont'd
11.4/37 cont'd

To find the intersection, plug the equations for \( L \) into the equation for \( P \):

\[
3(3+2t) + 2(6-5t) - 4(2+3t) = 1.
\]

\[
9 + 6t + 12 - 10t - 8 - 12t = 1
\]

\[
13 - 16t = 1, \quad -16t = -12, \quad t = \frac{12}{16} = \frac{3}{4}
\]

Using the equations for \( L \),

\[
\chi = 3 + 2(\frac{3}{4}) = 3 + \frac{3}{2} = \frac{9}{2}, \quad \text{(the point of intersection)}
\]

\[
y = 6 - 5(\frac{3}{4}) = 6 - \frac{15}{4} = \frac{9}{4}
\]

\[
z = 2 + 3(\frac{3}{4}) = 2 + \frac{9}{4} = \frac{17}{4}
\]

11.4/40 The normal vectors are

\[
\vec{n} = \langle 2, -1, 1 \rangle, \quad \vec{m} = \langle 1, 1, -1 \rangle
\]

\[
\vec{n} \cdot \vec{m} = 2 - 1 - 1 = 0, \quad \text{so} \quad \vec{n} \perp \vec{m}.
\]

The angle is 90° or \( \frac{\pi}{2} \) radians.
To find a point on the intersection of the two planes, let
\[ z = 0 \text{ (this is arbitrary - any } z \text{ works)} \]

Then \[ 2x - y = 5 \] and \[ x + y = 1 \].

Solving for \( x \) and \( y \), we get \( x = 2, \ y = -1 \)

So \( P_0 = (2, -1, 0) \) is on the line of intersection.

The direction vector \( \vec{d} \) is parallel to both planes, so it is perpendicular to both \( \vec{n} \) and \( \vec{m} \) (see #40 for notation).

We get it by taking \( \vec{n} \times \vec{m} \):

\[ \vec{n} \times \vec{m} = \langle 2, -1, 1 \rangle \times \langle 1, 1, -1 \rangle \]

\[ = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 2 & -1 & 1 \\ 1 & 1 & -1 \end{vmatrix} = \langle 3, 3, 3 \rangle \]

Parameteric eq's of line: \( x = 2, \ y = -1 + 3t, \ z = 0 + 3t \)

Symmetric eq's: \( x = 2, \ \frac{y + 1}{3} = \frac{z}{3} \)
Plan: ① Find the direction vector \( \vec{v} \) of the intersection of the given planes.
② Then \( \overrightarrow{PQ} \) and \( \vec{v} \) are both perpen. to the normal vector \( \vec{n} \) of the plane we seek, so \( \vec{n} = \overrightarrow{PQ} \times \vec{v} \). ③ Find the equation by using \( P \) and \( \vec{n} \).

① The normal vectors of the given planes are \( \vec{m}_1 = \langle 1, 1, 1 \rangle \), \( \vec{m}_2 = \langle 3, -1, 0 \rangle \). The direction vector \( \vec{v} \) of their intersection is
\[
\vec{v} = \vec{m}_1 \times \vec{m}_2 = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 1 & 1 \\ 3 & -1 & 0 \end{vmatrix} = \hat{i} \begin{vmatrix} 1 & 1 \\ -1 & 0 \end{vmatrix} - \hat{j} \begin{vmatrix} 1 & 1 \\ 3 & 0 \end{vmatrix} + \hat{k} \begin{vmatrix} 1 & 1 \\ 3 & -1 \end{vmatrix}
\]
\[
= \hat{i} + 3 \hat{j} - 4 \hat{k} = \langle 1, 3, -4 \rangle
\]

② \( \vec{n} = \overrightarrow{PQ} \times \vec{v} = \langle 1, 1, 1 \rangle \times \langle 1, 3, -4 \rangle 
\]
\[
= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 1 & 1 \\ 3 & -4 & 0 \end{vmatrix} = \hat{i} \begin{vmatrix} 1 & 1 \\ -4 & 0 \end{vmatrix} - \hat{j} \begin{vmatrix} 1 & 1 \\ 3 & 0 \end{vmatrix} + \hat{k} \begin{vmatrix} 1 & 1 \\ 3 & -4 \end{vmatrix}
\]
\[
= -7 \hat{i} + 5 \hat{j} + 2 \hat{k} = \langle -7, 5, 2 \rangle
\]

③ Equation of a plane containing \( P \) and perpendicular to \( \langle -7, 5, 2 \rangle = \vec{n} \) is
\[
-7(x-1) + 5y + 2(z+1) = 0
\]
11.4/52 Parametric equations for the lines:
\[ t = x - 1 \quad x = t + 1 \]
\[ t = \frac{1}{2} (y + 1) \quad y = 2t - 1 \quad\text{and}\quad s = \frac{1}{3} (y - 2) \quad y = 3s + 2 \]
\[ t = z - 2 \quad z = t + 2 \quad s = \frac{1}{2} (z - 4) \quad z = 2s + 4 \]

To find the point of intersection:
\[
\begin{cases}
  t + 1 = s + 2 & (1) \\
  2t - 1 = 3s + 2 & (2) \\
  t + 2 = 2s + y & (3)
\end{cases}
\]

Using (1) and (3), we get \( s = -1, \ t = 0 \). This works in (2) also.

Putting \( t = 0 \) into the parametric equations for the first line gives \((x, y, z) = (1, -1, 2)\) as the point of intersection.

The direction vectors of the two lines are
\[
\vec{V}_1 = \langle 1, 2, 1 \rangle, \quad \vec{V}_2 = \langle 1, 3, 2 \rangle.
\]

Then \( \vec{V}_1 \times \vec{V}_2 \) is the normal vector of the plane containing the two lines.
\[
\vec{V}_1 \times \vec{V}_2 = \begin{vmatrix}
  1 & 2 & 1 \\
  3 & 2 & 1 \\
  1 & 2 & 1 
\end{vmatrix}
= \vec{i} - \vec{j} + \vec{k} = \langle 1, -1, 1 \rangle
\]

The equation of the plane is
\[ 1 \cdot (x - 1) - 1 \cdot (y + 1) + 1 \cdot (z - 2) = 0 \quad\text{or}\quad \boxed{x - y + z = 4} \]