Math 221. AL1 - Test #2 - October 26, 2011

Name: ____________________________

Signature: ________________________

Circle your discussion section:  AD1 (9:00, Stephen)  AD2 (10:00, James)
AD3 (11:00, Nik)  AD4 (12:00, Susannah)  AD5 (1:00, Stephen)
AD6 (2:00, Nik)  AD7 (3:00, James)  AD8 (4:00, Susannah)

DO NOT OPEN EXAM UNTIL TOLD TO DO SO.
SIT IN THE SEAT CIRCLED BELOW.

114 DKH

FRONT OF ROOM

Time: 50 minutes. You may not use any books or notes or calculator. There are 100 points possible.
To get any credit, you must show your work. Unless indicated, you do not need to simplify your answers.
Partial credit will be based only on what is actually written on the paper. All intermediate steps should be correct as written. During the test, we will not answer questions about test problems. If a question seems unclear or wrong to you, please note this (in writing) on your exam paper.

<table>
<thead>
<tr>
<th>number</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>8</th>
<th>9</th>
<th>10</th>
</tr>
</thead>
<tbody>
<tr>
<td>points</td>
<td>6</td>
<td>5</td>
<td>18</td>
<td>10</td>
<td>5</td>
<td>6</td>
<td>10</td>
<td>10</td>
<td>15</td>
<td>15</td>
</tr>
</tbody>
</table>

score ____________________________
1. (6 points) Find the function \( f(x) \) with \( f''(x) = 12x^2 + 2 \) and \( f(0) = -2, \ f'(1) = 7 \).

\[
\begin{align*}
\frac{d^2}{dx^2} f(x) &= 4x^3 + 2x + C \\
7 &= 4 \cdot 1^3 + 2 \cdot 1 + C \\
C &= 7 - 6 = 1 \\
\frac{d}{dx} f(x) &= 4x^3 + 2x + 1 \\
f(x) &= x^4 + x^2 + x + D \\
-2 &= 0 + 0 + 0 + D \\
D &= -2 \\
f(x) &= x^4 + x^2 + x - 2
\end{align*}
\]

2. (5 points) Give the definition of *inflection point*.

A point at which the concavity of a function changes. (From up to down or from down to up).
3. (6 points each part) For each part, find \( f'(x) \).

(a) \( f(x) = e^{\sin 3x} \)

\[
f'(x) = e^{\sin 3x} \cdot (\cos 3x) \quad (3)
\]

(b) \( f(x) = (\cos x)^x \)

\[
\ln y = \ln (\cos x)^x = x \ln (\cos x)
\]

\[
\frac{dy}{dx} = \frac{dy}{dx} = \frac{d}{dx} (\ln (\cos x) + x \cos x (-\sin x))
\]

\[
\frac{dy}{dx} = y \left( \ln (\cos x) - x \frac{\sin x}{\cos x} \right)
\]

\[
f'(x) = (\cos x)^x \left( \ln (\cos x) - x \frac{\sin x}{\cos x} \right)
\]

(c) \( f(x) = \sin^{-1}(2x + 1) \)

\[
f'(x) = \frac{1}{\sqrt{1-(2x+1)^2}} \cdot 2
\]
4. (5 points each part) Find each limit. You may use any method. Show your work or briefly explain your reasoning – no credit given for the answer alone.

(a) \[
\lim_{x \to \infty} \frac{x^3}{\ln x} \quad \text{of the form} \quad \frac{\infty}{\infty}
\]
\[
= \lim_{x \to \infty} \frac{\frac{1}{3} x^{2/3}}{\frac{1}{x}} = \lim_{x \to \infty} \frac{x}{3 x^{2/3}}
\]
\[
= \lim_{x \to \infty} \frac{x^{1/3}}{3} = \infty
\]

(b) \[
\lim_{x \to \pi} \frac{1 - \sin x}{\cos x}
\]
\[
= \frac{1 - \sin \pi}{\cos \pi} = \frac{1 - 0}{-1} = -1
\]

Can't use l'Hospital's Rule
5. (5 points) State the Mean Value Theorem. Be sure to include hypothesis ("if") and conclusion ("then").

If \( f \) is continuous on \([a,b]\) and differentiable on \((a,b)\), then there is a point \( c \) in \((a,b)\) with

\[
 f'(c) = \frac{f(b) - f(a)}{b - a}
\]

6. (6 points) The graph of a function \( y = f(x) \) and a point \( x_1 \) are shown. Using Newton's method on \( f \) with initial approximation \( x_1 \), mark on the \( x \)-axis where \( x_2 \) will be and briefly explain how you found \( x_2 \). One sentence should be enough.

\( x_2 \) is the \( x \)-intercept of the tangent line to \( y = f(x) \) at \((x_1, f(x_1))\).
7. (10 points) A plane flying horizontally at an altitude of 1 mile and a speed of 500 mi/hr passes directly over a radar station. Find the rate at which the distance from the plane to the station is increasing when it is 2 miles away from the radar station.

Find \( \frac{dz}{dt} \) when \( z = 2 \).

Given \( \frac{dx}{dt} = 500 \text{ mi/hr} \).

\[
1^2 + x^2 = z^2
\]

\[
0 + 2x \frac{dx}{dt} = 2z \frac{dz}{dt}.
\]

When \( z = 2 \), \( 1 + x^2 = 2^2 \), so \( x = \sqrt{3} \)

\[
2\sqrt{3} \cdot 500 = 2 \cdot 2 \frac{dz}{dt}
\]

\[
250\sqrt{3} = \frac{dz}{dt}
\]

Answer: \( 250\sqrt{3} \text{ mi/hr} \).
8. (10 points) The half-life of cesium-137 is 30 years. Suppose we have a 100-mg. sample. Find an expression for the mass that remains after \( t \) years.

\[
m = m(0)e^{kt}
\]

Given \( m(0) = 100 \)

and \( m = \frac{1}{2}(100) = 50 \) when \( t = 30 \).

\[
m = 100e^{kt}
\]

\[50 = e^{k \cdot 30}\]

\[\ln 50 = k \cdot 30\]

\[\frac{\ln 50}{30} = k\]

\[
m = 100e^{\frac{\ln 50}{30}t}
\]
9. (5 points each part) The graph of \( y = f'(x) \) is given. Be sure to notice that the questions ask you about the function \( y = f(x) \).

(a) On what intervals is \( f \) increasing and on what intervals is \( f \) decreasing?

- Increasing approximately \((-2, 1)\)
- Decreasing \((-4, -2)\) and \((1, 3)\)

(b) On what intervals is \( f \) concave up and on what intervals is \( f \) concave down?

- Concave up approximately \((-3, 0)\)
- Concave down approximately \((-4, -3)\) and \((0, 3)\)

(c) At what \( x \)-value does \( f \) have the largest rate of increase?

At about \( x=0 \) (where \( f' \) is the largest).
10. (3 points each part) Answer true or false for each part. You do not need to show your work or give any reason, and there is no partial credit on this problem.

(a) If \( \lim_{x \to 0} f(x) = 0 \) and \( \lim_{x \to 0} g(x) = 0 \), then \( \lim_{x \to 0} f(x)g(x) = 0 \).

\[ \boxed{F} \quad \text{(it is indeterminate)} \]

(b) If \( f \) is continuous on the closed interval \([a, b]\), then \( f \) attains an absolute maximum value \( f(c) \) and an absolute minimum value \( f(d) \) at some numbers \( c \) and \( d \) in \([a, b]\).

\[ \boxed{T} \quad \text{(This is the Extreme Value Theorem)} \]

(c) If \( f \) has a critical point at \( x = c \), then \( f \) must have either a local maximum or a local minimum at \( x = c \).

\[ \boxed{F} \quad \text{(\( f(x) = x^3 \), \( c = 0 \), for example)} \]

(d) If \( f'(x) > 0 \) for all \( x \), then there is at most one \( c \) with \( f(c) = 0 \).

\[ \boxed{T} \quad \text{(by Mean Value Theorem)} \]

(e) The linear approximation (also called linearization) of \( f(x) = 3x^2 \) at \( a = 1 \) is \( L(x) = 6 + 3(x-1) \).

\[ \boxed{F} \quad \text{(should be \( L(x) = 3 + 6(x-1) \))} \]