Statistical mechanics of spin models and quantum models

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Inspired by problems in physics, I aim in my research to make rigorous connections between the microscopic models of interacting particles and the macroscopic descriptions of phenomena, part of Hilbert's sixth problem, making physics rigorous. My field of statistical mechanics is deep, challenging, and multi-faceted: I use and create a variety of tools in probability and analysis.

A big challenge in mathematical physics is explaining physical phenomena rigorously from first principles. Superconductors, for instance, have essentially zero resistivity at low temperatures, and magnetic fields can bend around them, allowing levitation and applications to particle accelerators and MRI machines. We would like to understand theories of superconductivity mathematically, following work of Leggett and others, but many of these challenges remain intractable to current mathematical methods.

These phenomena could be understood by scaling limits, analogous to considering a drop of dye in water: If we were to zoom in to the molecular scale, we would see the particles colliding with each other. Each particle goes in one direction for a bit, then hits another particle and goes in a new direction until the next collision. The particles' movements are Newtonian and sensitive to their initial position and velocity. Out at the centimeter scale, we see dye spreading out in water uniformly, behavior that is probabilistic and described by the diffusion (or heat) equation. Connecting the microscopic motion and the macroscopic behavior through a scaling limit is our goal. To achieve this, I plan to develop a wide variety of mathematical tools and to prove scaling limits.

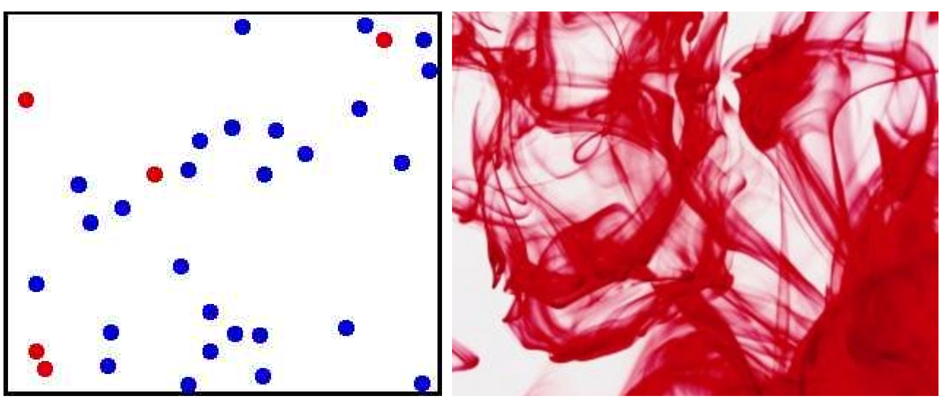

Figure 1: Left: Microscopic particles bounce off each other (courtesy of Greg L, Wikipedia). Right: Red dye diffuses in water on the macroscopic scale (Digital Vision/Getty Images).

One line of my research has been focused on low-temperature statistical mechanics, with applications to the ultra-cold quantum phenomenon called Bose-Einstein condensation (BEC), a phase of matter in which quantum particles called bosons can condense into the same one-particle state and behave as if they are one macroscopic quantum particle.

A physics heuristic of Gross and Pitaevskii in the 1960s suggested that the one-particle state of the BEC evolves according the cubic nonlinear Schrödinger equation (NLS), and this motivated a great deal of research in mathematical analysis. We have recently been able to make the rigorous connection between the physics of the interacting bosons and the macroscopic NLS.

My first work along these lines was collaboration with Schlein and Staffilani on two-dimensional quantum many-body systems in a localizing limit. We justify the cubic NLS as the macroscopic description of BEC in two dimensions, for both the plane and the torus, by developing a new approach for the most challenging part of the proof, a new approach using combinatorics as well as number theory in the spirit of Gauss.

The next questions are natural for a probabilist and important for developing quantum control theory in engineering: Can we prove a central limit theorem and large deviations principles for these observables? I answer the first and I have been working on the second, both in collaboration with Ben Arous and Schlein.

Anderson localization is also of interest in mathematical physics, and some of my work addresses this. For instance, we have found a mathematical explanation for the experimentally observed Anderson localization in BEC: we prove a phase transition in the Gibbs measures of the discrete NLS in dimensions three and higher, including a collaboration with Chatterjee. Our proof also suggests that certain solitary wave structures are stable and typical for the NLS, with interesting implications for questions about stability and typicality of solitary wave structures.

Another line of my research is on the statistical mechanics of spin models of superconductors, magnets, and the Higgs sector. Near zero temperature, the macroscopic model of superconductors is given by Bose-Einstein condensation and the nonlinear Schrödinger (NLS) equation. Superconductivity is often identified with the ferromagnetic phase in the XY model.

The classical XY, Heisenberg, and Toy Higgs models are examples of the general N-vector models, which we study in the mean-field limit, collaborations with Meckes and Nawaz.These models have a phase transition at a certain critical temperature, where an important physical quantity called magnetization (average spin) goes from Gaussian to non-Gaussian behavior. We have also found formulas for the free energy, descriptions of the canonical macrostates, and a second-order phase transition at the critical temperature.

There are other interesting physical quantities for general N-vector models, including susceptibility, specific heat, and correlation length. We will study these quantities, and I am interested in what we will discover and what this will mean for the spontaneous symmetry breaking in the Higgs sector model.

I will continue exploring similar challenges to improve our understanding of these phenomena. One long-term project (joint with Staffilani) is to understand the behavior of badly behaved nonlinear Schrödinger equations (NLS) in the spirit of my previous work. I also aim to sharpen our understanding of the critical temperature behavior of Gibbs measures in the 3D NLS, in joint work with Dey. And in a related project (joint with Chen and Dey), we are studying the Gibbs measures for the BBGKY hierarchy associated to spatially discretized quantum systems in 3D, in order to circumvent the notorious challenge of Gibbs measures for the badly behaved 3D NLS.

This work has applications to a number of areas in physics and engineering, specifically for developing quantum control theory. For instance, experimentalists hope to create quantum computers from BECs, but BECs are fragile and difficult to manipulate in laboratories, so it is important to understand them theoretically.