1. Prove that
\[ \limsup_{n} A_n = \{ x : x \text{ is in infinitely many } A_n \} \]  
(1)

2. Give an example of a set \( E \) and two \( \sigma \)-algebras, \( \mathcal{E}, \mathcal{F} \) on \( E \) such that \( \mathcal{E} \) is neither coarser nor finer than \( \mathcal{F} \).

3. Give examples of the following, all defined on \( \mathbb{Z} \):
   
   (a) A probability measure (also known as a distribution);
   (b) A finite measure that is not a distribution;
   (c) A \( \sigma \)-finite measure that is not finite;
   (d) A measure that is not \( \sigma \)-finite.

4. Let \( X \) be a random variable that takes values in \( \mathbb{N} \). Prove that
\[ \mathbb{E}[X] = \sum_{n=1}^{\infty} \mathbb{P}(X \geq n). \]

5. We define the function \( \varphi : \mathbb{N} \to \mathbb{N} \) as the number of distinct \( \sigma \)-algebras defined on the set \( S = \{1, 2, 3, \ldots, n\} \).
   
   (a) Let \( \psi(n) \) be the number of distinct collections of subsets of \( S \). Show that \( \psi(n) = 2^{2^n} \) and that \( \varphi(n) \leq \psi(n) \).
   (b) Show that for \( n > 1 \), \( \varphi(n+1) > \varphi(n) \), so that \( \varphi \) is a strictly increasing function.
   (c) Compute by hand \( \varphi(k) \) for \( k = 0, 1, 2, 3, 4 \) by enumerating all possible \( \sigma \)-algebras (of course, the theorem about the correspondence between sigma algebras and partitions will help). Do you see a pattern? Perhaps you might find this pattern in the Integer Sequence Database. How does the growth of this sequence compare to the bounds in part (a)?

6. Prove the Law of Total Probability, that if \( \{B_n\} \) is a partition of \( \Omega \), then
\[ \mathbb{P}(A) = \sum_{n} \mathbb{P}(A|B_n)\mathbb{P}(B_n). \]

7. **Expanded definition of RV:** Given a probability space \((\Omega, \mathcal{F}, \mathbb{P})\) and a measurable space \((E, \mathcal{E})\), an \((\mathcal{E}/\mathcal{F})\)-measurable random variable is a measurable function \( X : \Omega \to E \).
   
   Consider the function \( f : X \to Y \), where \( Y \) is a countable space. Choose the \( \sigma \)-algebra \( \mathcal{Y} = 2^Y \) and choose any \( \sigma \)-algebra \( \mathcal{X} \) on \( X \). Show that \( f \) is \((\mathcal{Y}/\mathcal{X})\)-measurable if and only if \( f^{-1}(i) \in \mathcal{X} \).
8. We will flip three fair coins, and use $\Omega = \{H, T\}^3$ as our probability space. We define $X$ as the total number of heads on the three flips, and define $\mathcal{F}_1$ as the information gained after one flip.

(a) Show that $\mathbb{E}[X | \mathcal{F}_1] : \Omega \to \mathbb{R}$ only takes two values, and determine the sets on which they are constant.

(b) Compute these two values using the theorem about linearity of expectation, by writing everything out explicitly.

9. Consider the same filtration as for the previous exercise: $\mathcal{F}_1, \mathcal{F}_2, \mathcal{F}_3$, generated by the information gained after one, two and three flips, respectively. Let $X$ be the random variable that counts whether or not an odd number of heads have been flipped, i.e. $X(\omega) = 1$ if $\omega$ has an odd number of $H$s, and 0 otherwise.

(a) Give an exact description of the random variable $\mathbb{E}[X | \mathcal{F}_2](\omega)$.

(b) Compute $\mathbb{E}[\mathbb{E}[X | \mathcal{F}_2]]$ by hand and check that it matches the Law of Total Expectation.