Sample Space, Events and Probability

Sample Space and Events

There are lots of phenomena in nature, like tossing a coin or tossing a die, whose outcomes cannot be predicted with certainty in advance, but the set of all the possible outcomes is known. These are what we call *random phenomena* or *random experiments*. Probability theory is concerned with such random phenomena or random experiments.

Consider a random experiment. The set of all the possible outcomes is called the *sample space* of the experiment and is usually denoted by S. Any subset E of the sample space S is called an *event*. Here are some examples.

Example 1 *Tossing a coin. The sample space is* $S = \{H, T\}$. $E = \{H\}$ *is an event.*

Example 2 *Tossing a die. The sample space is* $S = \{1, 2, 3, 4, 5, 6\}$. $E = \{2, 4, 6\}$ *is an event, which can be described in words as "the number is even".*

Example 3 *Tossing a coin twice. The sample space is* $S = \{HH, HT, TH, TT\}$. $E = \{HH, HT\}$ *is an event, which can be described in words as "the first toss results in a Heads.*

Example 4 *Tossing a die twice. The sample space is* $S = \{(i, j) : i, j = 1, 2, \ldots, 6\}$*, which contains 36 elements. "The sum of the results of the two toss is equal to 10" is an event.*

Example 5 *Choosing a point from the interval* $(0, 1)$. *The sample space is* $S = (0, 1)$. $E = (1/3, 1/2)$ *is an event.*

Example 6 *Measuring the lifetime of a lightbulb. The sample space is* $S = [0, \infty)$. $E = [90, \infty)$ *is an event.*

Example 7 *Keeping on tossing a coin until one gets a Heads. The sample space of this experiment is* $S = \{H, TH, TTH, TTTH, \ldots\}$. $E = \{H, TH\}$ *is an event.*

Suppose that E is an event. We say that the event E "occurs" if the outcome of the experiment is contained in E.

Since events are simply subsets of the sample space, we can talk about various set theoretic operations on events. In the following, E, F, G, E_i, $= 1, 2, \ldots$ are events.

$E \cup F$ denotes the union of E and F. $E \cap F$ denotes the intersection of E and F. E^c stands for the complement of E, that is $E = S \setminus E$. $E \subset F$ means that E is a subset of F. If $E \cap F = \emptyset$, we say that E and F are disjoint.

Similarly, we can talk about the union and intersection of more than two events:

$$\cup_{i=1}^{n} E_i, \quad \cup_{i=1}^{\infty} E_i, \quad \cap_{i=1}^{n} E_i, \quad \cap_{i=1}^{\infty} E_i.$$

Now we recall some properties of set theoretic operations:
Commutativity:

$$E \cup F = F \cup E, \quad E \cap F = F \cap E.$$

Associativity:

$$(E \cup F) \cup G = E \cup (F \cup G), \quad (E \cap F) \cap G = E \cap (F \cap G).$$

Distributivity:

$$(E \cup F) \cap G = (E \cap G) \cup (F \cap G), \quad (E \cap F) \cup G = (E \cup G) \cap (F \cup G).$$

De Morgan's law:

$$\begin{aligned} (\cup_{i=1}^n E_i)^c &= \cap_{i=1}^n E_i^c, \qquad (\cap_{i=1}^n E_i)^c = \cup_{i=1}^n E_i^c, \\ (\cup_{i=1}^\infty E_i)^c &= \cap_{i=1}^\infty E_i^c, \qquad (\cap_{i=1}^\infty E_i)^c = \cup_{i=1}^\infty E_i^c. \end{aligned}$$

Axioms of Probability

Consider an experiment with sample space S. A real-valued function $\mathbb{P}$ on the space of all events of the experiment is called a probability measure if

(i) for all events E, $0 \le \mathbb{P}(E) \le 1$;

(ii) $\mathbb{P}(S) = 1$;

(iii) for any sequence of events $E_1, E_2, \dots$ which are mutually disjoint

$$\mathbb{P}(\cup_{n=1}^\infty E_n) = \sum_{n=1}^\infty \mathbb{P}(E_n).$$

For any event E, we refer to $\mathbb{P}(E)$ as the probability of E.
Here are some examples.

Example 8 *Tossing a fair coin. In this case, the probability measure is given by* $\mathbb{P}(H) = \mathbb{P}(T) = \frac{1}{2}$. *If the coin is not fair, the probability measure will be different.*

Example 9 *Tossing a fair die. In this case, the probability measure is given by* $\mathbb{P}(1) = \mathbb{P}(2) = \cdots = \mathbb{P}(6) = \frac{1}{6}$. *If the die is not fair, the probability measure will be different.*

Example 10 *Tossing a fair coin twice. In this case, the probability measure is given by* $\mathbb{P}(HH) = \mathbb{P}(HT) = \mathbb{P}(TH) = \mathbb{P}(TT) = \frac{1}{4}$.

Example 11 *Tossing a fair die twice. In this case, the probability measure is given by*

$$\mathbb{P}((i,j)) = \frac{1}{36}, \quad i, j = 1, \dots, 6.$$

Example 12 *Tossing a point from* $\{1, \ldots, n\}$ *at random, that is each point is equally likely to be chosen. In this case, the probability measure is given by*

$$\mathbb{P}(i) = \frac{1}{n}, \quad i = 1, \ldots, n.$$

Example 13 *Choosing a point from the interval* (a, b) *at random, that is each point is equally likely to be chosen. In this case, the probability measure is given by*

$$\mathbb{P}((c,d)) = \frac{d-c}{b-a}, \qquad \textit{for all interval } (c,d) \subset (a,b).$$

Example 14 *Measuring the lifetime of a lightbulb. Depending on the manufacturer, the probability measure will be different. One possible probability measure is given by*

$$\mathbb{P}(E) = \int_E e^{-t} dt, \qquad \textit{for any } E \subset [0, \infty).$$

Example 15 *Keeping on tossing a fair coin until one gets a Heads. In this case, the probability measure is given by* $\mathbb{P}(H) = \frac{1}{2}, \mathbb{P}(TH) = \frac{1}{4}, \mathbb{P}(TTH) = \frac{1}{8}, \ldots$.

The following result list some properties of probability measures.

Theorem 16 *Suppose that* $\mathbb{P}$ *is a probability measure. Then it satisfies the following properties.*

(i) $\mathbb{P}(\emptyset) = 0$.

(ii) For any $n \geq 2$, *if* $E_1, \ldots, E_n$ *are disjoint events, then* $\mathbb{P}(\cup_{i=1}^n E_i) = \sum_{i=1}^n \mathbb{P}(E_i)$.

(iii) If $E \subset F$, *then* $\mathbb{P}(E) \leq \mathbb{P}(F)$.

(iv) For any event E, $\mathbb{P}(E^c) = 1 - \mathbb{P}(E)$.

(v) For any events E *and* F, $\mathbb{P}(E \cup F) = \mathbb{P}(E) + \mathbb{P}(F) - \mathbb{P}(E \cap F)$.

(vi) For any $n \geq 2$ *and any events* $E_1, \ldots, E_n$,

$$\mathbb{P}(\cup_{i=1}^n E_i) = \sum_{i=1}^n \mathbb{P}(E_i) - \sum_{i_1 < i_2} \mathbb{P}(E_{i_1} \cap E_{i_2}) + \cdots + (-1)^{k+1} \sum_{i_1 < \cdots < i_k} \mathbb{P}(\cap_{j=1}^k E_{i_j}) + \cdots + (-1)^{n+1} \mathbb{P}(\cap_{i=1}^n E_i).$$

The last formula is called the inclusion-exclusion formula.

Proof. We are only going to prove (i), (ii), (v). (vi) follows from (v) and the induction.

To prove (i), take $E_1 = S$, $E_2 = E_3 = \cdots = \emptyset$. Then $E_1, E_2, \ldots$ is a sequence of disjoint events, so we have

$$1 = \mathbb{P}(S) = \sum_{n=1}^{\infty} \mathbb{P}(E_n) = 1 + \mathbb{P}(\emptyset) + \mathbb{P}(\emptyset) + \cdots.$$

Consequently we have $\mathbb{P}(\emptyset) = 0$.

To prove (ii), take $E_{n+1} = E_{n+2} = \cdots = \emptyset$, then $E_1, E_2, \ldots$ is a sequence of disjoint events, so we have

$$\mathbb{P}(\cup_{i=1}^{n} E_i) = \mathbb{P}(\cup_{i=1}^{\infty} E_i) = \sum_{i=1}^{\infty} \mathbb{P}(E_i) = \sum_{i=1}^{n} \mathbb{P}(E_i).$$

To prove (v), notice that $E = (E \cap F) \cup (E \cap F^c)$, $F = (E \cap F) \cup (F \cap E^c)$ and $E \cup F = (E \cap F) \cup (E \cap F^c) \cup (F \cap E^c)$. Hence

$$\begin{aligned}
&\mathbb{P}(E) = \mathbb{P}(E \cap F) + \mathbb{P}(E \cap F^c),\\
&\mathbb{P}(F) = \mathbb{P}(E \cap F) + \mathbb{P}(F \cap E^c),\\
&P(E \cup F) = \mathbb{P}(E \cap F) + \mathbb{P}(E \cap F^c) + \mathbb{P}(F \cap E^c).
\end{aligned}$$

(v) follows immediately from the three identities above. □

Example 17 *A fair die is tossed twice. Find the probability that the sum of the two results is even.*

Solution. *Let E be the event that the sum of the two results is even. For $i = 2, 4, \cdots, 12$, let E_i be the event that the sum of the two results is i. Then $E_2, E_4, \cdots, E_{12}$ are disjoint and E is the union of $E_2, E_4, \cdots, E_{12}$. One can easily find the probability of each E_i, adding them up, we get $\mathbb{P}(E) = \frac{1}{2}$.*

Example 18 *A fair die is tossed 100 times. Find the probability that there is at least one 5.*

Solution. *Let E be the event that there is at least one 5. Then E^c is the event that there is no 5 and $\mathbb{P}(E^c) = (\frac{5}{6})^{100}$. Thus $\mathbb{P}(E) = 1 - (\frac{5}{6})^{100}$.*

Example 19 *Suppose that E and F are two events. If we know the probabilities of E, F and $E \cap F$, we can find the probability of any set theoretic combination of E and F. For instance, if $\mathbb{P}(E) = \frac{1}{2}$, $\mathbb{P}(F) = \frac{1}{3}$ and $\mathbb{P}(E \cap F) = \frac{1}{4}$, then*

$$\mathbb{P}(E \cup F) = \mathbb{P}(E) + \mathbb{P}(F) - \mathbb{P}(E \cap F) = \frac{1}{2} + \frac{1}{3} - \frac{1}{4}$$

and

$$\mathbb{P}(E \cap F^c) = \mathbb{P}(E) - \mathbb{P}(E \cap F) = \frac{1}{3} - \frac{1}{4}.$$