Dark Matter may be an axion Bose-Einstein condensate

Kay Kirkpatrick @kay314159

2021
Dark Matter may be an axion Bose-Einstein condensate

Kay Kirkpatrick, Urbana-Champaign, Miami and Mascouten lands

Figure: Compound image of some galaxies, one magnified (NRAO)

AMS Fall 2021 Central Sectional Meeting
Twitter: @kay314159
Sculpture of Sophie (or Sofya or Sonya) Kovalevskaya, 1850-1891
Big challenge: proving physics theorems

microscopic first principles $\rightsquigarrow$ zoom out $\rightsquigarrow$ Macroscopic states
Big challenge: proving physics theorems

microscopic first principles ~ zoom out ~ Macroscopic states

Courtesy Greg L and Digital Vision/Getty Images
Big challenge: proving physics theorems

microscopic first principles $\leadsto$ zoom out $\leadsto$ Macroscopic states

Courtesy Greg L and Digital Vision/Getty Images

Bigger challenge: studying a thing that doesn’t reflect light
Dark Matter interacts with luminous stuff via gravity

Small box: radio image of a galaxy ~8 billion light-years distant.
Main box: Hubble image of galaxy group ~5 bn light-years away with a DM core that magnifies the distant galaxy 6x. (NRAO)
Small box: distant galaxy. Main image: mid-distance 6x magnifier.
We think dark matter exists for many reasons

- Galaxies rotate too fast for their regular matter and outer stars should fly off (Zwicky 1930s, Rubin 1970s+)
- Gravitational micro/lensing is pervasive throughout the sky
- The cosmic microwave background radiation has fluctuations
- Et cetera
We think dark matter exists for many reasons

- Galaxies rotate too fast for their regular matter and outer stars should fly off (Zwicky 1930s, Rubin 1970s+)
- Gravitational micro/lensing is pervasive throughout the sky
- The cosmic microwave background radiation has fluctuations
- Et cetera

Regular matter is about 5% of the energy/matter in the cosmos; dark matter is likely about 25% and dark energy about 70%.

(Gravity may need a fix, but that can’t explain most DM evidence.)
Outline: DM as axion Bose-Einstein Condensates

1. Theories for Dark Matter and connection to BECs

2. Abstract results for BEC with NLS and Hartree equations

3. Theoretical predictions: BEC DM may actually exist right now
There are many theoretical physics models of dark matter.

Venn diagram of DM models, axions at lower left (credit: Tim M.P. Tait)
Peccei and Quinn (1970s) answered how QCD can preserve Charge+Parity symmetry by turning a parameter into a field.

Weinberg called it the “Higglet”
The Axion of Choice: a hypothetical pseudoscalar particle

Peccei and Quinn (1970s) answered how QCD can preserve Charge+Parity symmetry by turning a parameter into a field.

Weinberg called it the “Higglet”

Wilczek called it the “axion”
(it cleaned up a problem)
The equations for axion dark matter

A cubic nonlinear Schrödinger equation, NLS, aka Gross-Pitaevskii:

\[ i\partial_t \varphi = -\Delta \varphi/2m + \lambda |\varphi|^2 \varphi + U\varphi, \]

- \( m \) is the axion mass; \( \lambda > 0 \) for the focusing case
- \( U \): Newtonian gravitational potential for star-/galaxy-size DM
The equations for axion dark matter

A cubic nonlinear Schrödinger equation, NLS, aka Gross-Pitaevskii:

\[ i\partial_t \varphi = -\Delta \varphi/2m + \lambda |\varphi|^2 \varphi + U \varphi, \]

- \( m \) is the axion mass; \( \lambda > 0 \) for the focusing case
- \( U \): Newtonian gravitational potential for star-/galaxy-size DM

\[ \Delta U = 4\pi G m^2 (|\varphi|^2 - n). \]

- \( G \) is gravitational constant
- \( n \) is the mean density of the field \( \varphi \)

This is almost identical to the equation for BEC.
Bose-Einstein Condensation (BEC) is ultra cold and stays* coherent as if it were a single NLS quantum particle.

Rubidium atoms’ momenta concentrate at 50 nK. (Atomic Lab)

* terms and conditions apply
A quantum “particle” is really a wavefunction $\psi$

$\psi(x, t) \in L^2(\mathbb{R}^d)$ solves a Schrödinger equation:

$$i \partial_t \psi = -\Delta \psi + V_{\text{ext}}(x)\psi$$
A quantum “particle” is really a wavefunction $\psi$

$\psi(x, t) \in L^2(\mathbb{R}^d)$ solves a Schrödinger equation:

$$i\partial_t \psi = -\Delta \psi + V_{\text{ext}}(x)\psi =: H\psi$$

- $-\Delta = -\sum_{i=1}^{d} \partial_{x^i x^i} \geq 0$
- external trapping potential $V_{\text{ext}}$
- solution $\psi(x, t) = e^{-iHt}\psi_0(x)$
A quantum “particle” is really a wavefunction $\psi$

$\psi(x, t) \in L^2(\mathbb{R}^d)$ solves a Schrödinger equation:

$$i\partial_t \psi = -\Delta \psi + V_{\text{ext}}(x)\psi =: H\psi$$

- $-\Delta = -\sum_{i=1}^{d} \partial_{x^ix^i} \geq 0$
- external trapping potential $V_{\text{ext}}$
- solution $\psi(x, t) = e^{-iHt}\psi_0(x)$

$\int |\psi_0|^2 = 1 \implies |\psi(\cdot, t)|^2$ is a probability density for all $t$ (why?)
A “particle in a box” solves a linear Schrödinger equation

\[ i\partial_t \psi = -\Delta \psi + V_{\text{ext}}(x)\psi \quad \text{where} \quad V_{\text{ext}} = \text{“} \infty \cdot 1_{[0,1]^c} \text{”} \]
An $N$-particle wavefunction $\psi_N(x, t) =$

$$\psi_N(x_1, \ldots, x_N, t) \in L^2(\mathbb{R}^{dN})$$

solves an $N$-body equation:

$$i \partial_t \psi_N = \sum_{j=1}^{N} -\Delta x_j \psi_N + \sum_{i<j} W(x_i - x_j) \psi_N =: H_N \psi_N$$
An $N$-particle wavefunction $\psi_N(x, t) =$

$$\psi_N(x_1, \ldots, x_N, t) \in L^2(\mathbb{R}^{dN})$$ solves an $N$-body equation:

$$i \partial_t \psi_N = \sum_{j=1}^{N} -\Delta_{x_j} \psi_N + \sum_{i<j} W(x_i - x_j) \psi_N =: H_N \psi_N$$

- pair interaction potential $W$; external potential is zero
- solution $\psi_N(x, t) = e^{-iH_N t} \psi_N^0(x)$ but is impractical
- joint probability density $|\psi_N(x_1, \ldots, x_N, t)|^2$
Additionally, for $N$ bosons, $\psi_N$ is symmetric:

$$\psi_N(x_{\sigma(1)}, \ldots, x_{\sigma(N)}, t) = \psi_N(x_1, \ldots, x_N, t) \text{ if } \sigma \in S_N$$
Additionally, for $N$ bosons, $\psi_N$ is symmetric:

$$
\psi_N(x_{\sigma(1)}, \ldots, x_{\sigma(N)}, t) = \psi_N(x_1, \ldots, x_N, t) \quad \text{if } \sigma \in S_N
$$

Initial data factorized (i.i.d. particles):

$$
\psi^0_N(x) = \prod_{j=1}^{N} \varphi_0(x_j) \in L^2_s(\mathbb{R}^{3N}).
$$
Additionally, for $N$ bosons, $\psi_N$ is symmetric:

$$\psi_N(x_{\sigma(1)}, \ldots, x_{\sigma(N)}, t) = \psi_N(x_1, \ldots, x_N, t) \text{ if } \sigma \in S_N$$

Initial data factorized (i.i.d. particles):

$$\psi^0_N(x) = \prod_{j=1}^N \varphi_0(x_j) \in L^2_s(\mathbb{R}^{3N}).$$

But interactions create correlations for $t > 0$. 

Mean-field pair interaction $W = \frac{1}{N} V$

Weak: order $1/N$. Long distance: $V \in L^\infty(\mathbb{R}^3)$.

$$i\partial_t \psi_N = \sum_{j=1}^{N} -\Delta_{x_j} \psi_N + \frac{1}{N} \sum_{i<j}^{N} V(x_i - x_j) \psi_N.$$
Mean-field pair interaction $W = \frac{1}{N} V$

Weak: order $1/N$. Long distance: $V \in L^\infty(\mathbb{R}^3)$.

$$i \partial_t \psi_N = \sum_{j=1}^{N} -\Delta x_j \psi_N + \frac{1}{N} \sum_{i<j}^{N} V(x_i - x_j) \psi_N.$$ 

Spohn, 1980: If $\psi_N$ is initially factorized and approx. factorized for all $t$, i.e., $\psi_N(x, t) \simeq \prod_{j=1}^{N} \varphi(x_j, t)$, then $\psi_N \to \varphi$ in some sense and $\varphi$ solves the Hartree equation:

$$i \partial_t \varphi = - \Delta \varphi + (V \ast |\varphi|^2) \varphi.$$
Erdös and Yau, 2001: did not assuming approx. factorization and got convergence for Coulomb interaction $V(x) = 1/|x|$, 

$$i\partial_t \varphi = -\Delta \varphi + \left(\frac{1}{|.|} * |\varphi|^2\right) \varphi.$$
Other mean-field limit theorems

Erdös and Yau, 2001: did not assuming approx. factorization and got convergence for Coulomb interaction $V(x) = 1/|x|$, 

$$i \partial_t \varphi = - \Delta \varphi + \left( \frac{1}{|\cdot|} * |\varphi|^2 \right) \varphi.$$ 

Rodnianski-Schlein ’08 and others got a convergence rate:

$$\psi_N(\cdot, t) \to \varphi(\cdot, t) \quad \text{like} \quad \frac{Ce^{Kt}}{N}.$$
Erdös and Yau, 2001: did not assuming approx. factorization and got convergence for Coulomb interaction $V(x) = 1/|x|$, 

$$i\partial_t \varphi = -\Delta \varphi + \left( \frac{1}{|.|} \ast |\varphi|^2 \right) \varphi.$$ 

Rodnianski-Schlein '08 and others got a convergence rate:

$$\psi_N(\cdot, t) \to \varphi(\cdot, t) \text{ like } \frac{Ce^{Kt}}{N}.$$ 

Preview of localizing interactions: 

$$(V_N \ast |\phi|^2)\varphi \to (\delta \ast |\phi|^2)\varphi$$
Definition of BEC (at 0 K) extends factorized property

Almost all particles are nearly in the same state: one-particle marginals $\gamma_N^{(1)}$ converge in trace to the one-particle state $\varphi$

$$\gamma_N^{(1)} := \text{Tr}_{N-1} |\psi_N\rangle \langle \psi_N|$$
Definition of BEC (at 0 K) extends factorized property

Almost all particles are nearly in the same state: one-particle marginals $\gamma^{(1)}_N$ converge in trace to the one-particle state $\phi$

$$\gamma^{(1)}_N := \text{Tr}_{N-1} |\psi_N\rangle \langle \psi_N| \xrightarrow{N \to \infty} |\phi\rangle \langle \phi|$$
Definition of BEC (at 0 K) extends factorized property

Almost all particles are nearly in the same state: one-particle marginals $\gamma_N^{(1)}$ converge in trace to the one-particle state $\varphi$

$$\gamma_N^{(1)} := Tr_{N-1} |\psi_N\rangle\langle\psi_N| \xrightarrow{N \to \infty} |\varphi\rangle\langle\varphi|$$

$\psi_N \in L^2_2(\mathbb{R}^{3N})$ is Bose-Einstein condensed into $\varphi \in L^2(\mathbb{R}^3)$. Technical details: $|\varphi\rangle\langle\varphi|(x_1, x'_1) = \overline{\varphi}(x_1)\varphi(x'_1)$ and

$$\gamma_N^{(1)}(x_1; x'_1, t) := \int \overline{\psi}_N(x_1, x_{N-1}, t)\psi_N(x'_1, x_{N-1}, t)dx_{N-1}$$
BEC limit theorems with parameter $\beta \in (0, 1]$

Now localized strong interactions: $N^{d\beta} V(N^\beta(\cdot)) \rightarrow b\delta$.

$$H_N = \sum_{j=1}^{N} -\Delta x_j + \frac{1}{N} \sum_{i<j}^{N} N^{d\beta} V(N^\beta(x_i - x_j)).$$
BEC limit theorems with parameter $\beta \in (0, 1]$

Now localized strong interactions: $N^{d\beta} V(N^\beta(\cdot)) \to b\delta$.

$$H_N = \sum_{j=1}^{N} -\Delta x_j + \frac{1}{N} \sum_{i<j} N^{d\beta} V(N^\beta(x_i - x_j)).$$

Theorems (Erdös-Schlein-Yau 2006-2008 $d = 3$
K.-Schlein-Staffilani 2009 $d = 2$ plane and rational tori):
Systems that are initially BEC remain condensed for all time, and the macroscopic evolution is the NLS:

$$i \partial_t \varphi = -\Delta \varphi + b|\varphi|^2 \varphi.$$
A random variable/observable $A$ is an operator on $\mathcal{H}$. 

Quantum prob: Hilbert sp. $\mathcal{H}$, projections $\mathcal{P}$, state $\varphi$
A random variable/observable $A$ is an operator on $\mathcal{H}$.

The expectation of $A$ in a pure state $\varphi$ is

$$\mathbb{E}_\varphi[A] := \langle \varphi | A \varphi \rangle = \int \varphi(x) \overline{A \varphi(x)} dx.$$  

E.g., position observable $X(\varphi)(x) := x \varphi(x)$ with density $|\varphi|^2$. 

Quantum prob: Hilbert sp. $\mathcal{H}$, projections $\mathcal{P}$, state $\varphi$
Interference makes quantum probability tricky

 Courtesy of Jordgette
Quantum LLNs follow from the previous BEC theorems

If $A$ is a one-particle observable and

$$A_j = 1 \otimes \cdots \otimes 1 \otimes A \otimes 1 \otimes \cdots \otimes 1,$$

then $\forall \varepsilon > 0$ we have a law of large numbers, LLN:

$$\mathbb{P}_{\psi_N} \left\{ \left| \frac{1}{N} \sum_{j=1}^{N} A_j - \langle \varphi | A \varphi \rangle \right| \geq \varepsilon \right\} \xrightarrow{N \to \infty} 0.$$
The red shell-like remnants of a supernova (Credit: NASA)

Some BECs explode ...
The red shell-like remnants of a supernova (Credit: NASA)

Some BECs explode ... in a bosenova.
A supernova can be partly caused by dark matter. Control theory.
Central Limit Theorem for Coulomb interacting bosons

Theorem (Ben Arous, K., Schlein, 2013): Under suitable assumptions on $\psi^0_N$, $\varphi_0$, $A$, and $V$, then for each $t \in \mathbb{R}$,

$$A_t := \frac{1}{\sqrt{N}} \sum_{j=1}^{N} (A_j - \mathbb{E}_t A) \xrightarrow{\text{distrib. } \psi_N \text{ as } N \to \infty} \mathcal{N}(0, \sigma_t^2).$$
Central Limit Theorem for Coulomb interacting bosons

**Theorem (Ben Arous, K., Schlein, 2013):** Under suitable assumptions on $\psi_{N}^{0}$, $\varphi_{0}$, $A$, and $V$, then for each $t \in \mathbb{R}$,

$$A_{t} := \frac{1}{\sqrt{N}} \sum_{j=1}^{N} (A_{j} - \mathbb{E}_{\varphi_{t}A}) \xrightarrow{\text{distr. } \psi_{N} \text{ as } N \to \infty} \mathcal{N}(0, \sigma_{t}^{2}).$$

The guessable variance is correct at $t = 0$ only:

$$\sigma_{0}^{2} = \mathbb{E}_{\varphi_{0}}[A^{2}] - (\mathbb{E}_{\varphi_{0}}A)^{2}.$$

But $\sigma_{t}^{2}$ has $\varphi_{0} \leadsto \varphi_{t}$ and
Central Limit Theorem for Coulomb interacting bosons

**Theorem (Ben Arous, K., Schlein, 2013):** Under suitable assumptions on $\psi_N^0$, $\varphi_0$, $A$, and $V$, then for each $t \in \mathbb{R}$,

$$A_t := \frac{1}{\sqrt{N}} \sum_{j=1}^{N} (A_j - \mathbb{E}_{\varphi_t} A) \xrightarrow{\text{distrib. } \psi_N \text{ as } N \to \infty} \mathcal{N}(0, \sigma_t^2).$$

The guessable variance is correct at $t = 0$ only:

$$\sigma_0^2 = \mathbb{E}_{\varphi_0} [A^2] - (\mathbb{E}_{\varphi_0} A)^2.$$

But $\sigma_t^2$ has $\varphi_0 \leadsto \varphi_t$ and a Bogoliubov transform twist.
Our novelty is using the bosonic Bogoliubov transform

The correct variance is the guess twisted by $\Theta_{t,0}$:

$$\sigma^2_t = \|U_t A \phi_t + JV_t A \phi_t\|^2 - |\langle \phi_t | U_t A \phi_t + JV_t A \phi_t \rangle|^2,$$

where $Jf = \overline{f}$ and $U_t, V_t$ are linear maps such that:

$$U_t^* U_t - V_t^* V_t = 1, \quad U_t^* JV_t J = V_t^* JU_t J.$$

**Rademacher and Schlein, 2019 and 2020:**
CLTs for localizing and Gross-Pitaevskii limits.
Theorem (K., Rademacher, Schlein, 2020): Under suitable (stricter) assumptions on $\psi_N^0$, $\varphi_0$, $A$, and $V$, there is a constant $C$ depending on $\|\varphi_0\|_{H^4}$ only, such that:
Theorem (K., Rademacher, Schlein, 2020): Under suitable (stricter) assumptions on $\psi_0^N$, $\varphi_0$, $A$, and $V$, there is a constant $C$ depending on $\|\varphi_0\|_{H^4}$ only, such that:

$$\frac{1}{N} \log \mathbb{E}_{\psi_N, t} e^{\lambda \left[ \sum (A_j - \langle \varphi_t, A \varphi_t \rangle) \right]} \leq \frac{\lambda^2}{2} \alpha_t^2 + C \lambda^3 \|A\|^3 e^{C(M+1)|t|},$$

for all $\lambda \leq \|A\|^{-1} e^{-CMt}$, $M = \|V\|_1 + \|V\|_\infty$, $\|A\|$ a combo norm, and $\alpha_t$ the initial data norm for a weird diff-eq.
Theorems about BEC, summarized

\[ H_N = \sum_{j=1}^{N} -\Delta x_j + \frac{1}{N} \sum_{i<j}^{N} V_N(x_i - x_j) \]

\( N \)-body Schröd.

**micro** : \( \psi_0 \rightarrow \psi_N \)

factorized/BEC \( \downarrow \)

**MACRO** : \( \varphi_0 \rightarrow \varphi \)

**NLS evolution**

\[ i\partial_t \varphi = -\Delta \varphi + b|\varphi|^2 \varphi. \]
Theorems about BEC, summarized

\[ H_N = \sum_{j=1}^{N} -\Delta x_j + \frac{1}{N} \sum_{i<j}^{N} V_N(x_i - x_j) \]

\[ N \text{-body Schröd.} \]

\[ \text{micro : } \psi_0^N \rightarrow \psi_N \]

factorized/BEC \[ \downarrow \quad \downarrow \quad \text{various senses} \]

\[ \text{MACRO : } \varphi_0 \rightarrow \varphi \]

\[ \text{NLS evolution} \]

\[ i\partial_t \varphi = -\Delta \varphi + b|\varphi|^2 \varphi. \]

What can all of this say about axion dark matter?
As the early universe cooled and QCD symmetry broke...

Axions probably started gravitating together and forming clumps with soliton cores called miniclusters or Bose stars:

\[ i\partial_t \varphi = -\Delta \varphi / 2m + \lambda |\varphi|^2 \varphi + U\varphi, \]

\[ \Delta U = 4\pi Gm^2(|\varphi|^2 - n). \]
As the early universe cooled and QCD symmetry broke...

Axions probably started gravitating together and forming clumps with soliton cores called miniclusters or Bose stars:

\[ i \partial_t \varphi = -\Delta \varphi / 2m + \lambda |\varphi|^2 \varphi + U \varphi, \]

\[ \Delta U = 4\pi Gm^2(|\varphi|^2 - n). \]

How long did or do axion BECs take to condense?

With Anthony Mirasola and Chanda Prescod-Weinstein
Axion BECs have both gravity and self-interactions

Which is stronger, gravity or self-interactions?

\[ i \partial_t \varphi = -\Delta \varphi / 2m + \lambda |\varphi|^2 \varphi + U \varphi, \]

\[ \Delta U = 4\pi Gm^2 (|\varphi|^2 - n). \]
Axion BECs have both gravity and self-interactions

Which is stronger, gravity or self-interactions?

\[ i \partial_t \varphi = -\Delta \varphi / 2m + \lambda |\varphi|^2 \varphi + U \varphi, \]

\[ \Delta U = 4\pi G m^2 (|\varphi|^2 - n). \]

In a Bose star: gravity \( \gg \) self-interaction (\( \lambda \sim 10^{-48} \))

In a DM bosenova: gravity \( \ll \) self-interaction, I suspect
Much axion BEC work is in the limit at 0 Kelvin

At $\varepsilon$ K, there is high occupancy of the ground state, and the particles have high density and tiny mean free path.

We also observed that the axions’ de Broglie wavelength is large, especially compared to the mean free path.
Much axion BEC work is in the limit at 0 Kelvin

At $\varepsilon$ K, there is high occupancy of the ground state, and the particles have high density and tiny mean free path.

We also observed that the axions’ de Broglie wavelength is large, especially compared to the mean free path.

Wigner functions capture interference in wave ensembles:

$$f(x, p, t) := \int e^{-i p \cdot y} \langle \psi^*(x + y/2) \psi(x - y/2) \rangle$$

(angled brackets: average over random phase Gaussian)
The Wigner function for $\psi$ which solves GPP / NLS-Poisson,

$$f(x, p, t) := \int e^{-ip \cdot y} \langle \psi^* (x + y/2) \psi(x - y/2), \rangle$$

has a relaxation/condensation time scale based on

$$\ddot{f}(t) \sim \frac{f}{T^2_\lambda} + \frac{f}{T^2_G}$$

and estimates from observations, etc., give concrete times

$$T_\lambda \sim 10^{22} \text{sec} \gg T_G \sim 10^{17} \text{sec} \sim \text{current age of cosmos}$$
The workings of the cosmos

Probability, PDE, Algorithms

↓      ↑

Physics, Biology

Experimental axion searches: ADMX, ABRACADABRA
(expected masses of axions run from $10^{-22}$ to $10^{-5}$ eV)
Thanks: NSF CAREER, Simons Sabbatical Fellowship

arXiv:1009.5737 (CPAM), 1111.6999 (CMP), 1505.05137(PJM), 2007.07438 (PRD), 2010.13754 (Annals HP), more @kay314159
A distant galaxy warped and “melted” by a lens (NASA Hubble)
Excellent theoretical physics books

The Disordered Cosmos, by Chanda Prescod-Weinstein
The End of Everything, by Katie Mack
Theorem (K., Rademacher, Schlein, 2020): If the initial state is factorized $\psi_{N,0} = \varphi_0^\otimes N$ with normalized $\varphi_0 \in H^4(\mathbb{R}^3)$, $A$ bounded self-adjoint on $L^2$ with $\|\Delta A(1 - \Delta)^{-1}\|_{op} < \infty$, and $V \in L^1 \cap L^\infty$, then $\exists C$ depending on $\|\varphi_0\|_{H^4}$ only, s.t.: 
Theorem (K., Rademacher, Schlein, 2020): If the initial state is factorized $\psi_{N,0} = \varphi_0 \otimes^N$ with normalized $\varphi_0 \in H^4(\mathbb{R}^3)$, $A$ bounded self-adjoint on $L^2$ with $\|\Delta A(1 - \Delta)^{-1}\|_{\text{op}} < \infty$, and $V \in L^1 \cap L^\infty$, then $\exists C$ depending on $\|\varphi_0\|_{H^4}$ only, s.t.:

$$\frac{1}{N} \log \mathbb{E}_{\psi_{N,t}} e^{\lambda \left[ \sum (A_j - \langle \varphi_t, A\varphi_t \rangle) \right]} \leq \frac{\lambda^2}{2} \alpha_t^2 + C \lambda^3 \|A\|^3 e^{C(M+1)|t|},$$

for all $\lambda \leq \|A\|^{-1} e^{-CMt}$ and $M = \|V\|_1 + \|V\|_\infty$, and $\alpha_t$ is the initial data norm for a weird differential equation and $\|A\|$ is a combo operator norm.
Our CLT for interacting quantum many-body systems

**Theorem (Ben Arous, K., Schlein, 2013):** If the initial state is factorized \( \psi_N^0 = \varphi_0^\otimes N \) with normalized \( \varphi_0 \in H^1(\mathbb{R}^3) \), and \( A \) is compact self-adjoint on \( L^2(\mathbb{R}^3) \), and \( V \leq 1/|\cdot| \), then for \( t \in \mathbb{R} \)
Our CLT for interacting quantum many-body systems

**Theorem (Ben Arous, K., Schlein, 2013):** If the initial state is factorized $\psi^0_N = \varphi^N_0$ with normalized $\varphi_0 \in H^1(\mathbb{R}^3)$, and $A$ is compact self-adjoint on $L^2(\mathbb{R}^3)$, and $V \leq 1/|\cdot|$, then for $t \in \mathbb{R}$

$$A_t := \frac{1}{\sqrt{N}} \sum_{j=1}^{N} (A_j - \mathbb{E}\varphi_t A) \xrightarrow{\text{distrib. as } N \to \infty} \mathcal{N}(0, \sigma^2_t).$$

Distribution of $A_t$ from $\psi_N = \psi_{N,t}$. 

The obvious variance is correct at $t = 0$ only (i.i.d.): $\sigma^2_0 = \mathbb{E}\varphi_0 \left[ A^2 \right] - \left( \mathbb{E}\varphi_0 A \right)^2 = ||A\varphi_0||^2 - \langle \varphi_0 | A\varphi_0 \rangle^2$. 

The variance $\sigma^2_t$ is more subtle than replacing $\varphi_0$ by $\varphi_t$. 

Our CLT for interacting quantum many-body systems

**Theorem (Ben Arous, K., Schlein, 2013):** If the initial state is factorized $\psi_0^N = \varphi_0^N$ with normalized $\varphi_0 \in H^1(\mathbb{R}^3)$, and $A$ is compact self-adjoint on $L^2(\mathbb{R}^3)$, and $V \leq 1/|\cdot|$, then for $t \in \mathbb{R}$

$$\mathcal{A}_t := \frac{1}{\sqrt{N}} \sum_{j=1}^{N} (A_j - \mathbb{E}_{\varphi_t} A) \xrightarrow{\text{distrib. as } N \to \infty} \mathcal{N}(0, \sigma_t^2).$$

Distribution of $\mathcal{A}_t$ from $\psi_N = \psi_{N,t}$.

The obvious variance is correct at $t = 0$ only (i.i.d.):

$$\sigma_0^2 = \mathbb{E}_{\varphi_0}[A^2] - (\mathbb{E}_{\varphi_0} A)^2 = \|A\varphi_0\|^2 - \langle \varphi_0 | A\varphi_0 \rangle^2.$$
Our CLT for interacting quantum many-body systems

**Theorem (Ben Arous, K., Schlein, 2013):** If the initial state is factorized \( \psi_0^N = \varphi_0^N \) with normalized \( \varphi_0 \in H^1(\mathbb{R}^3) \), and \( A \) is compact self-adjoint on \( L^2(\mathbb{R}^3) \), and \( V \leq 1/|\cdot| \), then for \( t \in \mathbb{R} \)

\[
A_t := \frac{1}{\sqrt{N}} \sum_{j=1}^{N} (A_j - \mathbb{E}_{\varphi_t}A) \xrightarrow{\text{distrib. as } N \to \infty} \mathcal{N}(0, \sigma_t^2).
\]

Distribution of \( A_t \) from \( \psi_N = \psi_{N,t} \).

The obvious variance is correct at \( t = 0 \) only (i.i.d.):

\[
\sigma_0^2 = \mathbb{E}_{\varphi_0}[A^2] - (\mathbb{E}_{\varphi_0}A)^2 = ||A\varphi_0||^2 - \langle \varphi_0 | A\varphi_0 \rangle^2.
\]

The variance \( \sigma_t^2 \) is more subtle than replacing \( \varphi_0 \) by \( \varphi_t \).
Proof sketch: first moment of $A_t = \frac{1}{\sqrt{N}} \sum (A_j - \mathbb{E}_{\varphi_t} A)$

First moment goes to the normal thing:

$$\left| \mathbb{E}_{\psi_N} A_t \right| = \left| \frac{1}{\sqrt{N}} \sum_{j=1}^{N} \text{Tr} A(\gamma^{(1)}_N - |\varphi\rangle \langle \varphi|) \right|$$

Higher moments too: combinatorics and Bog. transform.
Proof sketch: first moment of $A_t = \frac{1}{\sqrt{N}} \sum (A_j - \mathbb{E}_{\varphi_t} A)$

First moment goes to the normal thing:

$$|\mathbb{E}_{\psi_N} A_t| = \left| \frac{1}{\sqrt{N}} \sum_{j=1}^{N} \text{Tr} A(\gamma^{(1)}_N - |\varphi\rangle\langle\varphi|) \right|$$

$$\leq \frac{||A||}{\sqrt{N}} \sum_{j=1}^{N} \text{Tr} \left| \gamma^{(1)}_N - |\varphi\rangle\langle\varphi| \right|$$

$$\lesssim \frac{||A|| \cdot Ne^{Kt}}{\sqrt{N}} \xrightarrow{N \to \infty} 0.$$ 

Higher moments too: combinatorics and Bog. transform.
The bosonic Fock space

Fock space: \[ \mathcal{F} = \bigoplus_{n \geq 0} L_s^2(\mathbb{R}^{3n}, dx_1 \ldots dx_n) \]

Inner product:
\[ \langle \psi, \Phi \rangle = \overline{\psi(0)}\varphi(0) + \sum_{n \geq 1} \langle \psi^{(n)}, \varphi^{(n)} \rangle. \]
The bosonic Fock space

Fock space: \[ \mathcal{F} = \bigoplus_{n \geq 0} L^2_s(\mathbb{R}^{3n}, dx_1 \ldots dx_n) \]

Inner product: \[ \langle \psi, \phi \rangle = \overline{\psi(0)} \phi(0) + \sum_{n \geq 1} \langle \psi^{(n)}, \phi^{(n)} \rangle . \]

Number-of-particles operator: \[ \mathcal{N}\{\psi^{(n)}\}_{n \geq 0} = \{n\psi^{(n)}\}_{n \geq 0}, \]
eigenvectors \{0, \ldots, 0, \psi^{(m)}, 0, \ldots\}.

Hamiltonian:
\[ \mathcal{H}_N^{(m)} = \sum_{j=1}^{m} -\Delta x_j + \frac{1}{N} \sum_{i<j}^{m} V(x_i - x_j) . \]
The bosonic Fock space

Fock space: \[ \mathcal{F} = \bigoplus_{n \geq 0} L^2_s(\mathbb{R}^{3n}, dx_1 \ldots dx_n) \]

Inner product: \[ \langle \Psi, \Phi \rangle = \overline{\psi(0)} \varphi(0) + \sum_{n \geq 1} \langle \psi^{(n)}, \varphi^{(n)} \rangle. \]

Number-of-particles operator: \( \mathcal{N}\{\psi^{(n)}\}_{n \geq 0} = \{n\psi^{(n)}\}_{n \geq 0} \), eigenvectors \( \{0, \ldots, 0, \psi^{(m)}, 0, \ldots\} \).

Hamiltonian: \[ \mathcal{H}_N^{(m)} = \sum_{j=1}^{m} -\Delta_{x_j} + \frac{1}{N} \sum_{i<j}^{m} V(x_i - x_j). \]

Then \( e^{-i\mathcal{H}_N t}\{0, \ldots, 0, \psi_N, 0, \ldots\} = \{0, \ldots, 0, e^{-i\mathcal{H}_N t}\psi_N, 0, \ldots\} \).

Advantage: Particle number not fixed.
Creation and annihilation operators

\[
(a^* (f) \psi)^{(n)} (x_1, \ldots, x_n) = \frac{1}{\sqrt{n}} \sum_{j=1}^{n} f(x_j) \psi^{(n-1)}(x_1, \ldots, x_{j-1}, x_{j+1}, \ldots, x_n)
\]

\[
(a (f) \psi)^{(n)} (x_1, \ldots, x_n) = \sqrt{n + 1} \int dx \; \overline{f(x)} \psi^{(n+1)}(x, x_1, \ldots, x_n).
\]
Creation and annihilation operators

\[
(a^*(f)\psi)^{(n)}(x_1, \ldots, x_n) = \frac{1}{\sqrt{n}} \sum_{j=1}^{n} f(x_j)\psi^{(n-1)}(x_1, \ldots, x_{j-1}, x_{j+1}, \ldots, x_n)
\]

\[
(a(f)\psi)^{(n)}(x_1, \ldots, x_n) = \sqrt{n + 1} \int dx f(x)\psi^{(n+1)}(x, x_1, \ldots, x_n).
\]

Canonical commutation relations: \([a(f), a^*(g)] = \langle f, g \rangle\) and \([a(f), a(g)] = [a^*(f), a^*(g)] = 0.\)
Creation and annihilation operators

\[
(a^*(f)\psi)^{(n)}(x_1, \ldots, x_n) = \frac{1}{\sqrt{n}} \sum_{j=1}^{n} f(x_j)\psi^{(n-1)}(x_1, \ldots, x_{j-1}, x_{j+1}, \ldots, x_n)
\]

\[
(a(f)\psi)^{(n)}(x_1, \ldots, x_n) = \sqrt{n + 1} \int dx \overline{f(x)}\psi^{(n+1)}(x, x_1, \ldots, x_n).
\]

Canonical commutation relations: \([a(f), a^*(g)] = \langle f, g \rangle\) and
\([a(f), a(g)] = [a^*(f), a^*(g)] = 0\).

Operator-valued distributions \(a_x, a_x^*\):

\[
a(f) = \int dx f(x) a_x, \quad \text{and} \quad a^*(f) = \int dx f(x) a_x^*.
\]

Hamiltonian (commutes w/ particle number op. \(N = \int dx a_x^* a_x\)):

\[
\mathcal{H}_N = \int dx \nabla_x a_x^* \nabla_x a_x + \frac{1}{2N} \int dxdy V(x - y) a_x^* a_y^* a_y a_x.
\]
Replacement for product states

Product state with $N$ particles all in state $\varphi$:

$$\{0, \ldots, 0, \varphi \otimes N, 0, \ldots\} = \frac{(a^*(\varphi))^N}{\sqrt{N!}} \Omega.$$ 

Here the vacuum vector is $\Omega = \{1, 0, 0, \ldots\}$. 
Replacement for product states

Product state with $N$ particles all in state $\varphi$:

$$\{0, \ldots, 0, \varphi \otimes^N, 0, \ldots \} = (a^*(\varphi))^N \frac{\sqrt{N!}}{\Omega}.$$

Here the vacuum vector is $\Omega = \{1, 0, 0, \ldots \}$.

Weyl operator $W(\varphi) = e^{(a^*(\varphi) - a(\varphi))}$ to make a coherent state, with all particles all in state $\varphi$:

$$W(\varphi)\Omega = e^{-\|\varphi\|^2/2} \sum_{j=0} \frac{a^*(\varphi)^j}{j!} \Omega = e^{-\|\varphi\|^2/2} \{1, \varphi, \frac{\varphi \otimes 2}{\sqrt{2!}}, \ldots, \frac{\varphi \otimes j}{\sqrt{j!}}, \ldots \}.$$
Replacement for product states

Product state with $N$ particles all in state $\varphi$:

$$\{0, \ldots, 0, \varphi \otimes N, 0, \ldots\} = \frac{(a^*(\varphi))^N}{\sqrt{N!}} \Omega.$$ 

Here the vacuum vector is $\Omega = \{1, 0, 0, \ldots\}$.

Weyl operator $W(\varphi) = e^{(a^*(\varphi) - a(\varphi))}$ to make a coherent state, with all particles all in state $\varphi$:

$$W(\varphi)\Omega = e^{-\|\varphi\|^2/2} \sum_{j=0}^{\infty} \frac{a^*(\varphi)^j}{j!} \Omega = e^{-\|\varphi\|^2/2} \{1, \varphi, \frac{\varphi \otimes 2}{\sqrt{2!}}, \ldots, \frac{\varphi \otimes j}{\sqrt{j!}}, \ldots\}.$$ 

With respect to this coherent state, $N$ is a Poisson($\|\varphi\|^2$) RV.
The fluctuation dynamics

Around the mean-field approximation $W(\sqrt{N}\varphi_t)\Omega$, fluctuations

$$\mathcal{U}_N(t; s) = W^*(\sqrt{N}\varphi_t)e^{-i\mathcal{H}_N(t-s)}W(\sqrt{N}\varphi_s),$$

with generator

Limiting dynamics $\mathcal{U}_\infty(t; s)$ has generator $\mathcal{L}_\infty(t)$ and is described by the Bogoliubov transformation.
The fluctuation dynamics

Around the mean-field approximation $W(\sqrt{N}\varphi_t)\Omega$, fluctuations

$\mathcal{U}_N(t; s) = W^*(\sqrt{N}\varphi_t)e^{-i\mathcal{H}_N(t-s)}W(\sqrt{N}\varphi_s),$

with generator

$\mathcal{L}_N(t) = \int dx \nabla_x a_x^* \nabla_x a_x + \int dx (V * |\varphi_t|^2)(x)a_x^*a_x$

$+ \frac{1}{2} \int dx dy V(x - y) (\varphi_t(x)\varphi_t(y)a_x^*a_y + \overline{\varphi}_t(x)\overline{\varphi}_t(y)a_xa_y)$

$+ \int dx dy V(x - y) \varphi_t(x)\overline{\varphi}_t(y)a_x^*a_y + o(1)$

$= \mathcal{L}_\infty(t) + o(1).$
The fluctuation dynamics

Around the mean-field approximation $W(\sqrt{N}\varphi_t)\Omega$, fluctuations

$$\mathcal{U}_N(t; s) = W^*(\sqrt{N}\varphi_t)e^{-i\mathcal{H}_N(t-s)}W(\sqrt{N}\varphi_s),$$

with generator

$$\mathcal{L}_N(t) = \int dx \nabla_x a_x^* \nabla_x a_x + \int dx (V * |\varphi_t|^2)(x)a_x^*a_x$$

$$+ \frac{1}{2} \int dxdy V(x - y) (\varphi_t(x)\varphi_t(y)a_x^*a_y^* + \overline{\varphi}_t(x)\overline{\varphi}_t(y)a_xa_y)$$

$$+ \int dxdy V(x - y) \varphi_t(x)\overline{\varphi}_t(y)a_x^*a_y + o(1)$$

$$= \mathcal{L}_\infty(t) + o(1).$$

Limiting dynamics $\mathcal{U}_\infty(t, s)$ has generator $\mathcal{L}_\infty(t)$ and is described by the Bogoliubov transformation.