

Homework 8 — Solutions

Chapter 5

Problem 37 Let X be uniformly distributed over $(-1, 1)$.

(a) $P\{|X| > \frac{1}{2}\} = P\{X > \frac{1}{2}\} + P\{X < -\frac{1}{2}\} = \frac{1}{2}$

(b) Let $Y = |X|$. If $y \in (0, 1)$, then $F_Y(y) = P\{Y \leq y\} = P\{-y \leq Y \leq y\} = y$, so that

$$f_Y(y) = \begin{cases} 1 & 0 < y < 1 \\ 0 & \text{otherwise} \end{cases}$$

Problem 39 Let X be exponential with $\lambda = 1$, and let $Y = \log X$. Then $F_Y(y) = P\{Y \leq y\} = P\{\log X \leq y\} = P\{X \leq e^y\} = 1 - e^{-e^y}$, so that

$$f_Y(y) = e^{y-e^y}.$$

Problem 40 Let X be uniform on $(0, 1)$, and $Y = e^X$. Then, for $1 < y < e$, $F_Y(y) = P\{Y \leq y\} = P\{e^X \leq y\} = P\{X \leq \log Y\} = \log Y$, so that

$$f_Y(y) = \begin{cases} \frac{1}{y} & 1 < y < e \\ 0 & \text{otherwise} \end{cases}$$

Problem 41 For any $r \in (-A, A)$, we have $F_R r = P\{R \leq r\} = P\{A \sin \theta \leq r\} = P\{\theta \leq \arcsin \frac{r}{A}\} = \frac{1}{\pi} \arcsin \frac{r}{A}$, so that

$$f_R(r) = \begin{cases} \frac{1}{\pi \sqrt{A^2 - r^2}} & r \in (-A, A) \\ 0 & \text{otherwise} \end{cases}$$

Chapter 6

Problem 2 (a)

$P\{X_1 = i, X_2 = j\}$	$j = 0$	$j = 1$	$P\{X_1 = i\}$
$i = 0$	$\frac{8}{13} \frac{7}{12} = \frac{14}{39}$	$\frac{8}{13} \frac{5}{12} = \frac{10}{39}$	$\frac{24}{39}$
$i = 1$	$\frac{5}{13} \frac{8}{12} = \frac{10}{39}$	$\frac{5}{13} \frac{4}{12} = \frac{5}{39}$	$\frac{15}{39}$
$P\{X_2 = j\}$	$\frac{24}{39}$	$\frac{15}{39}$	1

(b)

$$\begin{aligned}P\{X_1 = 0, X_2 = 0, X_3 = 0\} &= \frac{8}{13} \frac{7}{12} \frac{6}{11} = \frac{28}{143} \\P\{X_1 = 0, X_2 = 0, X_3 = 1\} &= \frac{8}{13} \frac{7}{12} \frac{5}{11} = \frac{70}{429} \\P\{X_1 = 0, X_2 = 1, X_3 = 0\} &= \frac{8}{13} \frac{5}{12} \frac{7}{11} = \frac{70}{429} \\P\{X_1 = 1, X_2 = 0, X_3 = 0\} &= \frac{5}{13} \frac{8}{12} \frac{7}{11} = \frac{70}{429} \\P\{X_1 = 0, X_2 = 1, X_3 = 1\} &= \frac{8}{13} \frac{5}{12} \frac{4}{11} = \frac{40}{429} \\P\{X_1 = 1, X_2 = 0, X_3 = 1\} &= \frac{5}{13} \frac{8}{12} \frac{4}{11} = \frac{40}{429} \\P\{X_1 = 1, X_2 = 1, X_3 = 0\} &= \frac{5}{13} \frac{4}{12} \frac{8}{11} = \frac{40}{429} \\P\{X_1 = 1, X_2 = 1, X_3 = 1\} &= \frac{5}{13} \frac{4}{12} \frac{3}{11} = \frac{5}{143}\end{aligned}$$

Problem 7 $P\{X_1 = i, X_2 = j\} = p^2(1-p)^{i+j}$

Problem 8 X, Y are jointly continuous with probability density function

$$f(x, y) = \begin{cases} c(y^2 - x^2)e^{-y} & -y \leq x \leq y, 0 < y < \infty \\ 0 & \text{otherwise.} \end{cases}$$

(a) Note that

$$\int \int_{\mathbb{R}^2} f(x, y) = \int_0^\infty \int_{-y}^y c(y^2 - x^2)e^{-y} dx dy = 8c,$$

so that $c = \frac{1}{8}$.

(b)

$$\begin{aligned}f_X(x) &= \frac{1}{8} \int_{|x|}^\infty (y^2 - x^2)e^{-y} dy = \frac{(|x| + 1)e^{-|x|}}{4} \\f_Y(y) &= \frac{1}{8} \int_{-y}^y (y^2 - x^2)e^{-y} dx = \frac{1}{6} y^3 e^{-y} \quad \text{for } y > 0\end{aligned}$$

(c) $E[X] = 0$ by symmetry.

Problem 9 Let X, Y be jointly continuous with joint density function $f(x, y) = \frac{6}{7}(x^2 + \frac{xy}{2})$ for $0 < x < 1, 0 < y < 2$.

(a)

$$\int_0^1 \int_0^2 x^2 + \frac{xy}{2} dy dx = \int_0^1 2x^2 + x dx = \frac{7}{6}$$

(b)

$$f_X(x) = \frac{6}{7}x(2x + 1) \quad \text{for } 0 < x < 1$$

(c)

$$P\{X > Y\} = \int_0^1 \int_0^x f(x, y) dy dx = \frac{15}{56}$$

(d)

$$P\left\{Y > \frac{1}{2} \mid X < \frac{1}{2}\right\} = \frac{P\{X < \frac{1}{2}, Y > \frac{1}{2}\}}{P\{X < \frac{1}{2}\}} = \frac{\int_{\frac{1}{2}}^2 \int_0^{\frac{1}{2}} f(x, y) dx dy}{\int_0^{\frac{1}{2}} f_X(x) dx} = 0.8625$$

(e)

$$E[X] = \int_0^1 x f_X(x) dx = \frac{5}{7}$$

(f)

$$E[Y] = \int_0^2 y \int_0^1 f(x, y) dx dy = \frac{8}{7}$$

Problem 10 Let X, Y be jointly distributed with density function $f(x, y) = e^{-(x+y)}$ for $0 \leq x < \infty, 0 \leq y < \infty$.

(a) $P\{X < Y\} = \frac{1}{2}$ by symmetry

(b) $P\{X < a\} = \int_0^a \int_0^\infty e^{-(x+y)} dy dx = 1 - e^{-a}$