The mean-field Heisenberg model: exact solvability and non-normal asymptotics

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Joint with Elizabeth Meckes (Case Western)

Superconductors were discovered accidentally

1911: In the process of liquifying helium at 2.2 K, Onnes noticed the resistivity of mercury vanishing at 4.2 K.

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Figure: Courtesy of Mike Jory.

Superconductors are more than just perfect conductors

1930s: Superconductors (SCs) exhibit the Meissner effect, expelling magnetic fields, which allows levitation (courtesy of U Fribourg).

Superconducting magnets are used in MRI machines, particle accelerators, and measuring the Planck constant.

We understand superconductors imperfectly

60s heuristic: Bardeen-Cooper-Schrieffer (BCS) theory to Ginzburg-Landau (GL) and to Bose-Einstein condensation (BEC)

Erdős, K., Schlein, Staffilani, . . . : quantum systems to BEC; BCS to static GL. . . .

micro models $\rightsquigarrow$ $\rightsquigarrow$ Macro GL, BEC

$\searrow$ $\nearrow$

Meso BCS

Outline: Mean-field models give first insights

Simple spin models

More realistic ones

Mean-field Heisenberg model of magnets

XY model of superconductors? New ideas?

Ising started modeling magnets (1925)

1D Ising model has spin configuration $\sigma = (\sigma_i)_{i=1}^n \in \{\pm 1\}^n =: \Omega$

and Hamiltonian energy

$$H(\sigma) = -\sum \sigma_i \sigma_{i+1}$$

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Two global minima, or ground states:

Gibbs measures are natural, with inverse temperature β

$$Z^{-1} e^{-\beta H(\sigma)} = Z^{-1} \exp\left\{\beta \sum \sigma_i \sigma_{i+1}\right\}$$

$\beta = 0 \implies e^{-\beta H(\sigma)}$ is uniform. Disordered.

$\downarrow$

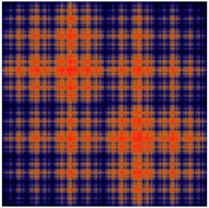
phase transition

$\downarrow$

$\beta = \infty \implies e^{-\beta H(\sigma)}$ picks out ground states. Ordered.

We know a lot about the Ising model, even in 2D

Onsager's solution above critical β:



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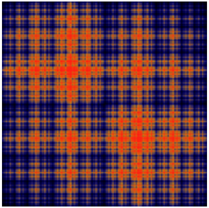

Hysteresis:

Movies thanks to Jack Weinstein.

But realistic models are harder

One way to increase dimension: Higher spin dimension

N-vector model: graph (V, E) with $n = |V|$ spins

spin configuration $\sigma \in (\mathbb{S}^{N-1})^n$

Hamiltonian energy

$$H_n(\sigma) = -\sum_{(i,j)\in E} J_{i,j} \langle \sigma_i, \sigma_j \rangle$$

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$N = 1$: Ising model (simplified magnet)
$N = 2$: XY model (superconductor)
$N = 3$: Heisenberg model (ferromagnet)

Another way: Higher lattice dimension

Mean-field model on graph $G = (V, E)$ with $|V| = n$ has

spin configuration $\sigma = (\sigma_i)_{i=1}^n \in (\mathbb{S}^{N-1})^n$ and Hamiltonian energy:

$$H_n(\sigma) = -\sum_{i,j} J_{i,j} \langle \sigma_i, \sigma_j \rangle .$$

1. Send $n \to \infty$ in complete graph $G = K_n$: mean-field interaction $J_{i,j} = \frac{1}{2n}\ \forall i, j$.

2. Send $d \to \infty$ in the d-dimensional lattice:

$$J_{i,j} = \begin{cases} J, & \textit{for neighbors } i, j \\ 0, & \textit{else}. \end{cases}$$

We know a lot about the mean-field Ising model

There is an interesting phase transition at $\beta_c = 1$.

- For $\beta < 1$, the average spin goes to zero (LLN), with a CLT.
- For $\beta > 1$, the average spin has normal asymptotics around two states.
- For $\beta = 1$, it has a non-normal limiting density $\propto e^{-x^4/12}$.

(Ellis-Newman '78 ... Chatterjee-Shao '11.)

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Mean-field N-vector models have phase transition at $\beta_c(N) = N$.

- For $\beta < \beta_c$, the average spin decays (Kesten-Schonmann '88)

We're just starting to understand the mean-field Heisenberg model (results with E. Meckes)

$$H_n(\sigma) = -\frac{1}{2n} \sum_{i,j=1}^{n} \langle \sigma_i, \sigma_j \rangle$$

- For the average spin $\frac{1}{n} \sum_{i=1}^{n} \sigma_i$, we have large deviation principles (LDPs) at any β.
- Analysis of the free energy recovers the phase transition at $\beta_c = 3$.
- We have limit theorems for the average spin above, below, and (most interestingly) at $\beta_c = 3$.
- The non-normal critical limiting density is $\propto t^5 e^{-3ct^2}$

We start with independent spins, $\beta = 0$

P_n is the product or uniform measure on $(\mathbb{S}^2)^n$.

Average spin $\frac{1}{n}\sum_{i=1}^{n}\sigma_i \xrightarrow{n\to\infty} 0$, LLN, and CLT.

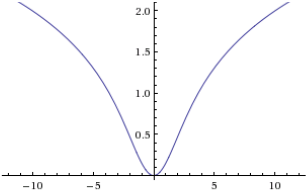
Theorem (K.-Meckes '13): Uniform random points $\{\sigma_i\}_{i=1}^n$ have a large deviation principle:

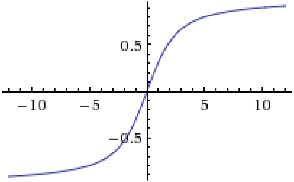
$$P_n\left(\frac{1}{n}\sum_{i=1}^{n}\sigma_i \simeq x\right) \simeq e^{-nI(x)},$$

where I is the rate function.

The rate function is a little obnoxious

I is implicitly





$$I(c) = cg(c) + \log\left(\frac{c}{\sinh(c)}\right), \qquad g(c) = \coth(c) - \frac{1}{c} = |x|.$$

Macrostates x are zeros of I: only $x = 0$ here. Disordered.

We go up to LDP level 2, $\beta = 0$

Empirical measure of spins: $\mu_{n,\sigma} = \frac{1}{n}\sum_{i=1}^{n} \delta_{\sigma_i}$

Theorem (K.-Meckes '13): We have a Sanov LDP:

$$P_n\{\mu_{n,\sigma} \in B\} \simeq \exp\{-n \inf_{\nu \in B} H(\nu|\mu)\}$$

where

$$H(\nu \mid \mu) := \begin{cases} \int_{\mathbb{S}^2} f \log(f) d\mu, & f := \frac{d\nu}{d\mu} \text{ exists;} \\ \infty, & \text{otherwise.} \end{cases}$$

Uniform measure μ and Borel subset B in $M_1(\mathbb{S}^2)$.

The only macrostate is μ, with $H(\mu \mid \mu) = 0$.

We transform level 2 to $\beta > 0$

Gibbs measures $P_{n,\beta}$ have densities $Z^{-1} e^{-\beta H_n(\sigma)}$.

Partition function: $Z = Z_n(\beta) = \int_{(\mathbb{S}^2)^n} e^{-\beta H_n(\sigma)} dP_n$.

Theorem (K.-Meckes '13): LDP w.r.t. Gibbs measures:

$$P_{n,\beta}\{\mu_{n,\sigma} \in B\} \simeq \exp\{-n \inf_{\nu \in B} I_\beta(\nu)\},$$

where

$$I_\beta(\nu) = H(\nu \mid \mu) - \frac{\beta}{2} \left| \int_{\mathbb{S}^2} x d\nu(x) \right|^2 - \varphi(\beta),$$

Zeros of I_β?

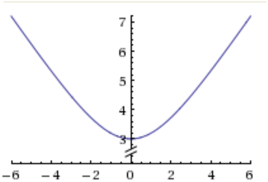
The free energy is obnoxious

$$\varphi(\beta) := -\lim_{n\to\infty}\frac{1}{n}\log Z_n(\beta) = \inf_{\nu}\left[H(\nu \mid \mu) - \frac{\beta}{2}\left|\int_{\mathbb{S}^2} x d\nu(x)\right|^2\right].$$

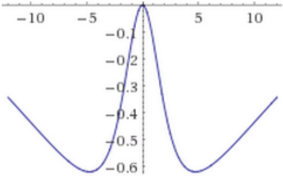
$$\varphi(\beta) = \begin{cases} 0, & \text{if } \beta < 3, \\ \Phi_\beta(\gamma^{-1}(\beta)), & \text{if } \beta \geq 3, \end{cases}$$

$$\Phi_\beta(k) := \log\left(\frac{k}{\sinh k}\right) + k\coth k - 1 - \frac{\beta}{2}\left(\coth k - \frac{1}{k}\right)^2$$

$\gamma(k) := \frac{k}{\coth k - 1/k} = \beta$



Φ_6



The phase transition and the macrostates

φ and φ' are continuous at $\beta_c = 3$ (2nd order phase transition)

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If $\beta < 3$, the macrostate (zero of I_β) is uniform.

If $\beta > 3$, the macrostates are rotations of the density

$$(x, y, z) \mapsto ce^{kz}, \text{ where } c = \frac{k}{2\sinh k}, \; k = \gamma^{-1}(\beta).$$

If $\beta \to \infty$, then $ce^{kz} \to \delta_{(0,0,1)}$, consistent with heuristic.

The average spin has a CLT below β_c

Theorem (K.-Meckes '13): For $\beta < 3$,

$$W_n := \sqrt{\frac{3-\beta}{n}} \sum_{i=1}^{n} \sigma_i \xrightarrow{distr.} Z,$$

Z is a standard normal random vector in $\mathbb{R}^3$.

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In the paper: There exists c_β such that

$$\sup_{h: M_1(h), M_2(h) \leq 1} |\mathbb{E}h(W_n) - \mathbb{E}h(Z)| \leq \frac{c_\beta \log(n)}{\sqrt{n}}$$

- $M_1(h)$ is the Lipschitz constant of h
- $M_2(h)$ is the maximum operator norm of the Hessian of h

The average spin has a CLT above β_c

Theorem (K.-Meckes '13): In the ordered phase, $\beta > 3$,

$$W_n := \sqrt{n}\left[\frac{\beta^2}{n^2k^2}\left|\sum_{j=1}^{n}\sigma_j\right|^2 - 1\right] \xrightarrow{distr.} Y,$$

Y is Gaussian with mean 0 and obnoxious variance

$\sigma^2 := \frac{4\beta^2}{(1-\beta g'(k))k^2}\left[\frac{1}{k^2} - \frac{1}{\sinh^2(k)}\right]$, for $g(x) = \coth x - \frac{1}{x}$.

(Or bounded-Lipschitz distance with an explicit rate of convergence.)

The limit is non-normal at $\beta_c = 3$

Theorem (K.-Meckes '13):

$$W_n := \frac{C}{n^{3/2}} \left| \sum_{j=1}^{n} \sigma_j \right|^2 \xrightarrow{distr.} X,$$

where X has density

$$p(t) = \begin{cases} \frac{1}{z} t^5 e^{-3ct^2} & t \geq 0; \\ 0 & t < 0, \end{cases}$$

with $c = \frac{1}{5C}$ and normalizing factor z.

Now we have definitive behavior of the mean-field Heisenberg model, especially at the critical temperature

- LDP methods, Ellis-Haven-Turkington method for $\beta > 0$
- Stein's method and a special non-normal version at β_c (Exchangeable pair via Glauber dynamics.)

- Connection to Schrödinger map and harmonic map heat flow?
- What about critical asymptotics for the mean-field XY model? PhD student Tayyab Nawaz is working on this.

The XY model describes superconductors, but not all

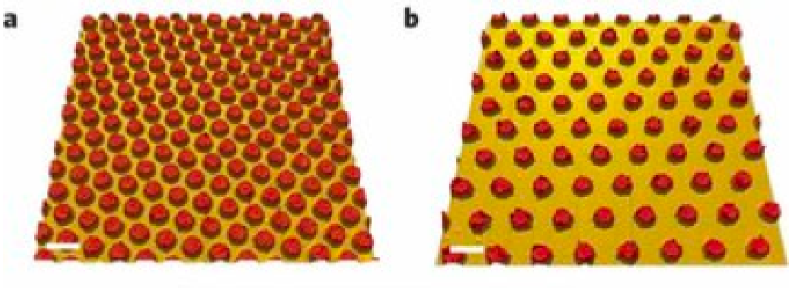


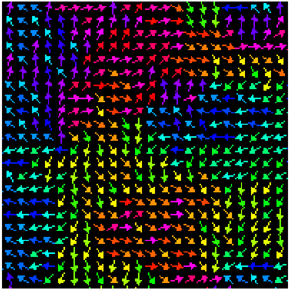
Figure: Red Nb islands on gold substrate, spaced 140nm & 340nm.

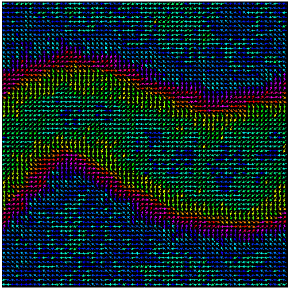
Nadya Mason's lab: two-step transition to superconductivity and zero-temperature metallic states.

The 2D XY model has hysteresis and metastability

On a torus, the Hamiltonian is:

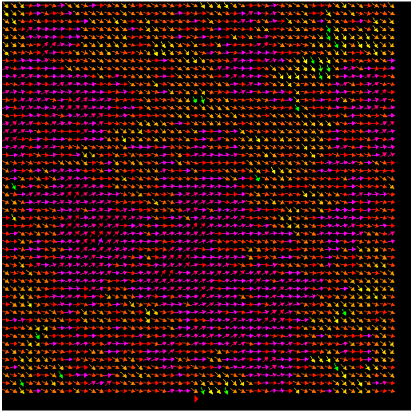
$$H(\sigma) = -\sum_{(i,j)\in E} \cos(\theta_i - \theta_j) - h\sum_{i\in V} \cos(\theta_i).$$





Batrouni '04 described "twisted" states like this.

We found more metastable states for the XY model



Topological classification of metastable states? (J. Weinstein)

What is being done and what could be done

- Dynamics between XY metastable states (L. DeVille)
- Two-step phase transition in XY chain (I. Maccari, A. Maiorano, E. Marinari, J. Weinstein)
- Is the zero-temperature metallic state a spin-glass?
- Critical asymptotics for mean-field XY?
- 3D Heisenberg? Schrödinger map, harmonic map heat flow

Thanks

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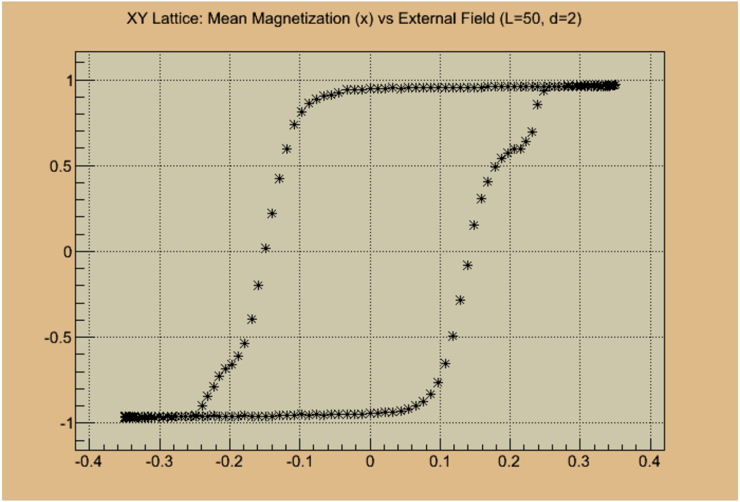
arXiv 1204.3062 (JSP)

A funny hysteresis curve for the XY model



Bumps correspond to loops or twisted states that a strong enough external field overcomes.