Ninth Homework Set — Solutions
Chapter 6

Problem 13 Let $X$ be uniform on $(-15, 15)$, and let $Y$ be uniform on $(-30, 30)$. Nobody waits longer than five minutes if $|Y - X| < 5$.

$$ P\{|Y - X| < 5\} = P\{-5 < Y - X < 5\} = P\{X - 5 < Y < X + 5\} = \int_{-15}^{15} \int_{x-5}^{x+5} \frac{1}{30 \cdot 60} dy \, dx = \frac{10 \cdot 30}{30 \cdot 60} = \frac{1}{6}.$$  

The probability that the man arrives first is $P\{X < Y\} = \frac{1}{2}$ by symmetry.

Problem 14 Let $X, Y$ be uniform random variables on $(0, L)$. Let $Z = |Y - X|$. We want to find $E[Z]$. First, find $F_Z(a)$, for $a \geq 0$. We have $F_Z(a) = P\{Z \leq a\} = P\{|Y - X| \leq a\} = P\{-a \leq Y - X \leq a\} = \frac{2aL - a^2}{L^2}$. Using geometric considerations. Hence, $f_Z(x) = \frac{2L - 2x}{L^2}$ if $0 \leq a \leq L$. Hence,

$$ E[Z] = \int_0^L x \cdot \frac{2L - 2x}{L^2} \, dx = \frac{2}{L^2} \left( \frac{Lx^2}{2} - \frac{x^3}{3} \right) \bigg|_0^L = \frac{L}{3}. $$

Problem 20 If the joint density function of $X$ and $Y$ is

$$ f(x, y) = \begin{cases} xe^{-(x+y)} & x > 0, y > 0 \\ 0 & \text{otherwise}, \end{cases} $$

then $f(x, y) = f_X(x)f_Y(y)$, where $f_X(x) = xe^{-x}$ for $x > 0$, and $f_Y(y) = e^{-y}$ for $y > 0$ (0 otherwise), so that $X$ and $Y$ are independent. If

$$ f(x, y) = \begin{cases} 2 & 0 < x < y, 0 < y < 1 \\ 0 & \text{otherwise}, \end{cases} $$
then $X$ and $Y$ are not independent because the nonzero values of $f$ are located in a triangular domain.

Problem 21

(a) Check: 
\[
\int_0^1 \int_0^{1-y} 24xy \, dx \, dy = \int_0^1 12(1-y)^2y \, dy = 12 \int_0^1 y - 2y^2 + y^3 \, dy = 6y^2 - 8y^3 + 3y^4|_0^1 = 6 - 8 + 3 = 1.
\]

(b) First, find 
\[
f_X(x) = \int_0^{1-x} 24xy \, dy = 24x(1-x)^2.
\]
Now, 
\[
E[X] = \int_0^1 12x^2(1-x)^2 \, dx = 4x^2 - 6x^3 + \frac{12}{5}x^5|_0^1 = 4 - 6 + \frac{12}{5} = \frac{2}{5}.
\]

(c) 
\[
E[Y] = E[X] = \frac{2}{5}
\]
by symmetry.

Problem 22

Let $X$ and $Y$ be jointly continuous with density function
\[
f(x,y) = \begin{cases} 
x + y & 0 < x < 1, 0 < y < 1 \\
0 & \text{otherwise}
\end{cases}
\]

(a) $X$ and $Y$ are not independent, since $f(x,y)$ is clearly not a product of functions of $x$ and $y$.

(b) $f_X(x) = \int_0^1 x + y \, dy = x + \frac{y^2}{2}|_0^1 = x + \frac{1}{2}$.

(c) $P\{X + Y < 1\} = \int_0^1 \int_0^{1-y} x + y \, dx \, dy = \int_0^1 (1-y)^2 \, dy = \frac{1}{2} \int_0^1 1 - y^2 \, dy = \frac{1}{2} (1 - \frac{1}{3}) = \frac{1}{3}$.

Problem 23

Let $X$ and $Y$ be jointly distributed with density function
\[
f(x,y) = \begin{cases} 
12xy(1-x) & 0 < x < 1, 0 < y < 1 \\
0 & \text{otherwise}
\end{cases}
\]

First, compute 
\[
f_X(x) = \int_0^1 12xy(1-x) \, dy = 6x(1-x) \quad \text{and} \quad f_Y(y) = \int_0^1 12xy(1-x) \, dx = 2y.
\]

(a) Clearly, $f(x,y) = f_X(x)f_Y(y)$, so that $X$ and $Y$ are independent.

(b) $E[X] = \int_0^1 6x^2(1-x) \, dx = 2x^3 - \frac{3}{2}x^4|_0^1 = \frac{1}{2}$.

(c) $E[Y] = \int_0^1 2y^2 \, dy = \frac{2}{3}y^3|_0^1 = \frac{2}{3}$.

(d) First, find 
\[
E[X^2] = \int_0^1 6x^2(1-x) \, dx = \frac{3}{2}x^4 - \frac{8}{5}x^5|_0^1 = \frac{3}{10}.
\]
Now, 
\[
\text{Var}(X) = E[X^2] - EX^2 = \frac{3}{10} - \frac{1}{4} = \frac{1}{20}.
\]

(e) First, find 
\[
E[Y^2] = \int_0^1 2y^3 \, dy = \frac{1}{2}y^4|_0^1 = \frac{1}{2}. \quad \text{Now, Var}(X) = \frac{1}{2} - \frac{1}{9} = \frac{1}{18}.
\]
Problem 27 Let $X$ be uniform on $(0, 1)$, and let $Y$ be exponential with $\lambda = 1$.

(a) If $Z = X + Y$, then

$$f_Z(a) = \int_{-\infty}^{\infty} f_Y(a - x)f_X(x)dx = \int_0^1 f_Y(a - x)dx$$

$$= \begin{cases} 0 & a < 0 \\ f_0^a e^{a-x}dx & 0 < a < 1 \\ f_0^1 e^{a-x}dx & 1 < a \end{cases}$$

$$= \begin{cases} 0 & a < 0 \\ 1 - e^{-a} & 0 < a < 1 \\ e^{-a}(e - 1) & 1 < a \end{cases}$$

(b) If $Z = \frac{X}{Y}$, we first find the cumulative distribution function of $Z$. Note that $F_Z(a) = 0$ if $a \leq 0$. Assume that $a > 0$.

$$F_Z(a) = P \{Z \leq a\} = P \{X \leq aY\}$$

$$= \int_0^1 \int_{\frac{x}{a}}^{\infty} f_Y(y)dydx$$

$$= a \left(1 - e^{-\frac{1}{a}}\right).$$

Now, we have

$$f_Z(a) = \frac{d}{da}F_Z(a) = \begin{cases} 0 & a \leq 0 \\ 1 - e^{-\frac{1}{a}} \left(1 + \frac{1}{a}\right) & 0 < a \end{cases}$$

Problem 28 Let $X_1, X_2$ be exponential random variables with parameter $\lambda_1, \lambda_2$. Let $Z = \frac{X_1}{X_2}$. Note that $F_Z(a) = 0$ if $a \leq 0$. Compute $F_Z(a)$ for $a > 0$:

$$F_Z(a) = P \{Z \leq a\} = P \{X_1 \leq aX_2\}$$

$$= \lambda_1\lambda_2 \int_0^{\infty} \int_0^{ag} e^{-\lambda_1x - \lambda_2y}dxdy$$

$$= \frac{\lambda_1a}{\lambda_1a + \lambda_2}.$$
so that 
\[ f_Z(a) = \frac{d}{da} F(a) = \frac{\lambda_1}{\lambda_1 a + \lambda_2} - \frac{\lambda_1^2 a}{(a\lambda_1 + \lambda_2)^2}. \]

Finally, we have 
\[ P\{X_1 < X_2\} = P\{Z < 1\} = F_Z(1) = \frac{\lambda_1}{\lambda_1 + \lambda_2}. \]

**Problem 29** Let \(X_1, X_2\) be independent normal random variables with \(\mu = 2200\) and \(\sigma^2 = 230^2\), representing the gross sales over this week and next week, respectively. Then \(X = X_1 + X_2\) is normal with mean 4400 and variance \(2 \cdot 230^2 = 105800\).

(a) \(P\{X > 5000\} = P\left\{ \frac{X - 4400}{\sqrt{105800}} > \frac{600}{\sqrt{105800}} \right\} = 1 - \Phi(1.84) = 1 - 0.9671 = 0.0329.\)

(b) Let \(p = P\{X_1 > 2000\} = P\left\{ \frac{X_1 - 2200}{230} > -\frac{200}{230} \right\} = 1 - \Phi(-\frac{20}{23}) = \Phi(0.87) = 0.8078.\)

Let \(N\) be the number of weeks (out of three) in which the sales exceed $2000. Then \(N\) is binomial with parameters \((p, 3)\), so that \(P\{N \geq 2\} = p^3 + 3p^2(1-p) = 0.9034\).

**Problem 33** Let \(X_1\) be the number of accidents in the next month, \(X_2\) the number of accidents in the month after that, and \(X_3\) the number of accidents in the third month. It makes sense to think of \(X_1, X_2,\) and \(X_3\) as independent Poisson random variables with parameter \(\lambda = 2.2\).

Let \(X = X_1, Y = X_1 + X_2,\) and \(Z = X_1 + X_2 + X_3\). Then \(X, Y,\) and \(Z\) are Poisson with parameter 2.2, 4.4, and 6.6, respectively.

(a) \(P\{X > 2\} = 1 - e^{-2.2} \left( 1 + 2.2 + \frac{2.2^2}{2} \right) = 0.3773.\)

(b) \(P\{Y > 4\} = 1 - e^{-4.4} \left( 1 + 4.4 + \frac{4.4^2}{2} + \frac{4.4^3}{3} + \frac{4.4^4}{4} \right) = 0.4488.\)

(c) \(P\{Z > 5\} = 1 - e^{-6.6} \left( 1 + 6.6 + \frac{6.6^2}{2} + \frac{6.6^3}{3} + \frac{6.6^4}{4} + \frac{6.6^5}{5} \right) = 0.6453.\)