HW 5 — Solutions
Chapter 3

Problem 64 Let $E$ be the event that the wife answers correctly, and let $F$ be the event that the husband answers correctly.

(a) If only one of them answers, then the probability of a correct answer is $P(E) = P(F) = p$.

(b) $P(\text{correct answer}) = P(EF) + \frac{1}{2} \cdot 2 \cdot p(1 - p) = p^2 + p - p^2 = p$

Problem 66 Let $E_i$ be the event that the $i$-th switch is on.

(a)

$P(\text{current flows from } A \text{ to } B) = (P(E_1E_2) + P(E_3E_4) - P(E_1E_2E_3E_4))P(E_5)$

$= (p_1p_2 + p_3p_4 - p_1p_2p_3p_4)p_5$

(b)

$P(\text{current flows from } A \text{ to } B) = P(E_1E_4 \cup E_1E_3E_5 \cup E_2E_5 \cup E_2E_3E_4)$

$= p_1p_4 + p_1p_3p_5 + p_2p_5 + p_2p_3p_4$

$- p_1p_3p_4p_5 - p_1p_2p_4p_5 - p_1p_2p_3p_4$

$- p_1p_2p_3p_5 - p_1p_2p_3p_4p_5 - p_2p_3p_4p_5$

$+ 4p_1p_2p_3p_4p_5 - p_1p_2p_3p_4p_5$

$= p_1p_4 + p_1p_3p_5 + p_2p_5 + p_2p_3p_4$

$- p_1p_3p_4p_5 - p_1p_2p_4p_5 - p_1p_2p_3p_4$

$- p_1p_2p_3p_5 - p_2p_3p_4p_5 + 2p_1p_2p_3p_4p_5$

Problem 78  (a) $P(\text{exactly four games are played}) = P(ABAA) + P(BAAA) + P(ABB) + P(BAB) = 2p^3(1 - p) + 2p(1 - p)^3 = 2p(1 - p)(p^2 + (1 - p)^2) = 2p(1 - p)(1 - 2p + 2p^2)$

(b) Let $E$ be the event that $A$ wins the match. Conditioning on the first two games of the match, we get $P(E) = P(E|A, A)p^2 +$
\[

because \( P(E|A, B) = P(E|B, A) = P(E) \).

Hence, \( P(E) = \frac{p^2}{1 - 2p(1 - p)} \).

Problem 81 Using the gambler’s ruin formula, the answer is

\[
\frac{1 - \left(\frac{9}{11}\right)^{15}}{1 - \left(\frac{9}{11}\right)^{30}}.
\]

Problem 83 (a) Conditioning on the coin flip

\[
P(\text{throw n is red}) = \frac{1}{2} \cdot \frac{6}{6} + \frac{1}{2} \cdot \frac{6}{6} = \frac{1}{2}.
\]

(b)

\[
P(R_3|R_1R_2) = \frac{P(R_1R_2R_3)}{P(R_1R_2)} = \frac{\frac{1}{2}(\frac{2}{3})^3 + \frac{1}{2}(\frac{1}{3})^3}{\frac{1}{2}(\frac{2}{3})^2 + \frac{1}{2}(\frac{1}{3})^2} = \frac{3}{5}.
\]

(c)

\[
P(A|R_1R_2) = \frac{P(R_1R_2|A)P(A)}{P(R_1R_2)} = \frac{(\frac{2}{3})^2 \cdot \frac{1}{2}}{\left(\frac{2}{3}\right)^2 \cdot \frac{1}{2} + \left(\frac{1}{3}\right)^2 \cdot \frac{1}{2}} = \frac{4}{5}.
\]

Problem 84 (a)

\[
P(A \text{ win}) = \sum_{i=0}^{\infty} P(\text{ the first white appears on draw number } 3i + 1)
\]

\[
= \sum_{i=0}^{\infty} \left(\frac{2}{3}\right)^{3i} \cdot \frac{1}{3} = \frac{1}{3} \sum_{i=0}^{\infty} \left(\frac{8}{27}\right)^i = \frac{1}{3} \cdot \frac{1}{1 - \frac{8}{27}}
\]

\[
P(B \text{ win}) = \sum_{i=0}^{\infty} P(\text{ the first white appears on draw number } 3i + 2)
\]

\[
= \sum_{i=0}^{\infty} \left(\frac{2}{3}\right)^{3i+1} \cdot \frac{1}{3} = \frac{2}{9} \sum_{i=0}^{\infty} \left(\frac{8}{27}\right)^i = \frac{2}{9} \cdot \frac{1}{1 - \frac{8}{27}}
\]

\[
P(C \text{ win}) = \sum_{i=0}^{\infty} P(\text{ the first white appears on draw number } 3i + 3)
\]

\[
= \sum_{i=0}^{\infty} \left(\frac{2}{3}\right)^{3i+2} \cdot \frac{1}{3} = \frac{4}{27} \sum_{i=0}^{\infty} \left(\frac{8}{27}\right)^i = \frac{4}{27} \cdot \frac{1}{1 - \frac{8}{27}}
\]
(b)

\[
P(A \text{ win}) = \frac{4}{12} + \frac{8}{12} + \frac{7}{12} + \frac{6}{12} + \frac{4}{12} + \frac{8}{12} + \frac{7}{12} + \frac{6}{12} + \frac{4}{12} + \frac{8}{12} + \frac{7}{12} + \frac{6}{12} + \frac{4}{12} + \frac{8}{12} + \frac{7}{12} + \frac{6}{12} + \frac{4}{12}
\]

\[
P(B \text{ win}) = \frac{8}{12} + \frac{4}{12} + \frac{8}{12} + \frac{7}{12} + \frac{6}{12} + \frac{5}{12} + \frac{9}{12} + \frac{8}{12} + \frac{7}{12} + \frac{6}{12} + \frac{5}{12} + \frac{9}{12} + \frac{8}{12} + \frac{7}{12} + \frac{6}{12} + \frac{5}{12}
\]

\[
P(C \text{ win}) = \frac{8}{12} + \frac{7}{12} + \frac{4}{12} + \frac{8}{12} + \frac{7}{12} + \frac{6}{12} + \frac{5}{12} + \frac{4}{12} + \frac{8}{12} + \frac{7}{12} + \frac{6}{12} + \frac{5}{12} + \frac{9}{12} + \frac{8}{12} + \frac{7}{12} + \frac{6}{12} + \frac{5}{12}
\]

Chapter 4

Problem 1 Possible values of \(X\): 0, 2, 4, -1, -2, 1

Probabilities:

\[
P\{X = 0\} = \frac{\binom{2}{2}}{\binom{14}{2}} = \frac{1}{91}
\]

\[
P\{X = 2\} = \frac{\binom{4}{2}}{\binom{14}{2}} = \frac{8}{91}
\]

\[
P\{X = 4\} = \frac{\binom{2}{2}}{\binom{14}{2}} = \frac{6}{91}
\]

\[
P\{X = -1\} = \frac{\binom{8}{2}}{\binom{14}{2}} = \frac{16}{91}
\]

\[
P\{X = -2\} = \frac{\binom{8}{2}}{\binom{14}{2}} = \frac{28}{91}
\]

\[
P\{X = 1\} = \frac{\binom{8}{1}}{\binom{14}{2}} = \frac{32}{91}
\]
Problem 4

\[ P\{X = 1\} = \frac{\binom{5}{1} 9!}{10!} = \frac{1}{2} \]

\[ P\{X = 2\} = \frac{\binom{5}{2} \binom{1}{1} 8!}{10!} = \frac{5}{18} \]

\[ P\{X = 3\} = \frac{\binom{5}{3} 2! \binom{1}{1} 7!}{10!} = \frac{5}{36} \]

\[ P\{X = 4\} = \frac{\binom{5}{4} 3! \binom{1}{1} 6!}{10!} = \frac{5}{84} \]

\[ P\{X = 5\} = \frac{\binom{5}{5} 4! \binom{1}{1} 4!}{10!} = \frac{1}{252} \]

\[ P\{X = 6\} = \frac{\binom{5}{5} 5! \binom{1}{1} 4!}{10!} = \frac{1}{252} \]

\[ P\{X = 7\} = P\{X = 8\} = P\{X = 9\} = P\{X = 10\} = 0 \]

Problem 5 The possible values are \( n, n-2, n-4, \ldots, -n+4, -n+2, -n \).

Problem 13 Let \( X \) be the total dollar value of all sales. Then \( X \) can take the values 0, 500, 1000, 1500, 2000, and we have

\[ P\{X = 0\} = 0.7 \cdot 0.4 = 0.28 \]

\[ P\{X = 500\} = \frac{1}{2} (0.3 \cdot 0.4 + 0.7 \cdot 0.6) = 0.27 \]

\[ P\{X = 1000\} = \frac{1}{2} (0.3 \cdot 0.4 + 0.7 \cdot 0.6) + \frac{1}{4} 0.3 \cdot 0.6 = 0.315 \]

\[ P\{X = 1500\} = \frac{1}{4} 0.3 \cdot 0.6 = 0.09 \]

\[ P\{X = 2000\} = \frac{1}{4} 0.3 \cdot 0.6 = 0.045 \]
Problem 14

\[ P \{ X = 0 \} = \frac{0!}{2!} = \frac{1}{2} \]
\[ P \{ X = 1 \} = \frac{1!}{3!} = \frac{1}{6} \]
\[ P \{ X = 2 \} = \frac{2!}{4!} = \frac{1}{12} \]
\[ P \{ X = 3 \} = \frac{3!}{5!} = \frac{1}{20} \]
\[ P \{ X = 4 \} = \frac{4!}{5!} = \frac{1}{5} \]

Problem 17 (a)

\[ P \{ X = 1 \} = P \{ X \leq 1 \} - P \{ X < 1 \} = \frac{1}{2} - \frac{1}{4} = \frac{1}{4} \]
\[ P \{ X = 2 \} = \frac{11}{12} - \frac{3}{4} = \frac{1}{6} \]
\[ P \{ X = 3 \} = 1 - \frac{11}{12} = \frac{1}{12} \]

(b) \[ P \{ \frac{1}{2} < X < \frac{3}{2} \} = \frac{5}{8} - \frac{1}{8} = \frac{1}{2} . \]

Problem 19

\[ P \{ X = 0 \} = \frac{1}{2} \]
\[ P \{ X = 1 \} = \frac{3}{5} - \frac{1}{2} = \frac{1}{10} \]
\[ P \{ X = 2 \} = \frac{4}{5} - \frac{3}{5} = \frac{1}{5} \]
\[ P \{ X = 3 \} = \frac{9}{10} - \frac{4}{5} = \frac{1}{10} \]
\[ P \{ X = 3.5 \} = 1 - \frac{9}{10} = \frac{1}{10} \]