HW 5 — Solutions
Chapter 3

Problem 64 Let E be the event that the wife answers correctly, and let F be the event that the husband answers correctly.

(a) If only one of them answers, then the probability of a correct answer is $P(E) = P(F) = p$.

(b) $P(\text{correct answer}) = P(EF) + \frac{1}{2} \cdot 2 \cdot p(1-p) = p^2 + p - p^2 = p$

Problem 66 Let E_i be the event that the i-th switch is on.

(a)

$$\begin{aligned}
&P(\text{current flows from } A \text{ to } B)\\
&= (P(E_1E_2) + P(E_3E_4) - P(E_1E_2E_3E_4))\, P(E_5)\\
&= (p_1p_2 + p_3p_4 - p_1p_2p_3p_4)p_5
\end{aligned}$$

(b)

$$\begin{aligned}
&P(\text{current flows from } A \text{ to } B)\\
&= P(E_1E_4 \cup E_1E_3E_5 \cup E_2E_5 \cup E_2E_3E_4)\\
&= p_1p_4 + p_1p_3p_5 + p_2p_5 + p_2p_3p_4\\
&-p_1p_3p_4p_5 - p_1p_2p_4p_5 - p_1p_2p_3p_4\\
&-p_1p_2p_3p_5 - p_1p_2p_3p_4p_5 - p_2p_3p_4p_5\\
&+4p_1p_2p_3p_4p_5 - p_1p_2p_3p_4p_5\\
&= p_1p_4 + p_1p_3p_5 + p_2p_5 + p_2p_3p_4\\
&-p_1p_3p_4p_5 - p_1p_2p_4p_5 - p_1p_2p_3p_4\\
&-p_1p_2p_3p_5 - p_2p_3p_4p_5 + 2p_1p_2p_3p_4p_5
\end{aligned}$$

Problem 78 (a) $P(\text{exactly four games are played}) = P(ABAA) + P(BAAA) + P(ABBB) + P(BABB) = 2p^3(1-p) + 2p(1-p)^3 = 2p(1-p)\left(p^2 + (1-p)^2\right) = 2p(1-p)(1-2p+2p^2)$

(b) Let E be the event that A wins the match. Conditioning on the first two games of the match, we get $P(E) = P(E|A,A)\, p^2 +$

$P(E|A,B)\,p(1-p) + P(E|B,A)\,(1-p)p + P(E|B,B)\,(1-p)^2 = p^2 + 2P(E)\,p(1-p)$ because $P(E|A,B) = P(E|B,A) = P(E)$. Hence, $P(E) = \frac{p^2}{1-2p(1-p)}$.

Problem 81 Using the gambler's ruin formula, the answer is

$$\frac{1-(9/11)^{15}}{1-(9/11)^{30}}.$$

Problem 83 (a) Conditioning on the coin flip

$$P(\text{throw n is red}) = \frac{1}{2}\frac{4}{6} + \frac{1}{2}\frac{2}{6} = \frac{1}{2}.$$

(b)

$$P(R_3|R_1R_2) = \frac{P(R_1R_2R_3)}{P(R_1R_2)} = \frac{\frac{1}{2}(\frac{2}{3})^3 + \frac{1}{2}(\frac{1}{3})^3}{\frac{1}{2}(\frac{2}{3})^2 + \frac{1}{2}(\frac{1}{3})^2} = \frac{3}{5}.$$

(c)

$$P(A|R_1R_2) = \frac{P(R_1R_2|A)P(A)}{P(R_1R_2)} = \frac{(\frac{2}{3})^2\frac{1}{2}}{(\frac{2}{3})^2\frac{1}{2} + (\frac{1}{3})^2\frac{1}{2}} = \frac{4}{5}.$$

Problem 84 (a)

$$\begin{aligned}
P(A\text{ win}) &= \sum_{i=0}^{\infty} P(\text{ the first white appears on draw number } 3i+1) \\
&= \sum_{i=0}^{\infty} (\frac{2}{3})^{3i}\frac{1}{3} = \frac{1}{3}\sum_{i=0}^{\infty}(\frac{8}{27})^i = \frac{1}{3}\frac{1}{1-\frac{8}{27}} \\
P(B\text{ win}) &= \sum_{i=0}^{\infty} P(\text{ the first white appears on draw number } 3i+2) \\
&= \sum_{i=0}^{\infty} (\frac{2}{3})^{3i+1}\frac{1}{3} = \frac{2}{9}\sum_{i=0}^{\infty}(\frac{8}{27})^i = \frac{2}{9}\frac{1}{1-\frac{8}{27}} \\
P(C\text{ win}) &= \sum_{i=0}^{\infty} P(\text{ the first white appears on draw number } 3i+3) \\
&= \sum_{i=0}^{\infty} (\frac{2}{3})^{3i+2}\frac{1}{3} = \frac{4}{27}\sum_{i=0}^{\infty}(\frac{8}{27})^i = \frac{4}{27}\frac{1}{1-\frac{8}{27}}
\end{aligned}$$

(b)

$$\begin{aligned}
P(A \text{ win}) &= \frac{4}{12} + \frac{8}{12}\frac{7}{11}\frac{6}{10}\frac{4}{9} + \frac{8}{12}\frac{7}{11}\frac{6}{10}\frac{5}{9}\frac{4}{8}\frac{3}{7}\frac{4}{6} \\
P(B \text{ win}) &= \frac{8}{12}\frac{4}{11} + \frac{8}{12}\frac{7}{11}\frac{6}{10}\frac{5}{9}\frac{4}{8} + \frac{8}{12}\frac{7}{11}\frac{6}{10}\frac{5}{9}\frac{4}{8}\frac{3}{7}\frac{2}{6}\frac{4}{5} \\
P(C \text{ win}) &= \frac{8}{12}\frac{7}{11}\frac{4}{10} + \frac{8}{12}\frac{7}{11}\frac{6}{10}\frac{5}{9}\frac{4}{8}\frac{4}{7} + \frac{8}{12}\frac{7}{11}\frac{6}{10}\frac{5}{9}\frac{4}{8}\frac{3}{7}\frac{2}{6}\frac{1}{5}.
\end{aligned}$$

Chapter 4

Problem 1 Possible values of X: 0,2,4,-1,-2,1 Probabilities:

$$P\{X=0\} = \frac{\binom{2}{2}}{\binom{14}{2}} = \frac{1}{91}$$

$$P\{X=2\} = \frac{\binom{4}{1}\binom{2}{1}}{\binom{14}{2}} = \frac{8}{91}$$

$$P\{X=4\} = \frac{\binom{4}{2}}{\binom{14}{2}} = \frac{6}{91}$$

$$P\{X=-1\} = \frac{\binom{8}{1}\binom{2}{1}}{\binom{14}{2}} = \frac{16}{91}$$

$$P\{X=-2\} = \frac{\binom{8}{2}}{\binom{14}{2}} = \frac{28}{91}$$

$$P\{X=1\} = \frac{\binom{8}{1}\binom{4}{1}}{\binom{14}{2}} = \frac{32}{91}$$

Problem 4

$$
\begin{aligned}
P\{X=1\} &= \frac{\binom{5}{1}9!}{10!} = \frac{1}{2} \\
P\{X=2\} &= \frac{\binom{5}{1}\binom{5}{1}8!}{10!} = \frac{5}{18} \\
P\{X=3\} &= \frac{\binom{5}{2}2!\binom{5}{1}7!}{10!} = \frac{5}{36} \\
P\{X=4\} &= \frac{\binom{5}{3}3!\binom{5}{1}6!}{10!} = \frac{5}{84} \\
P\{X=5\} &= \frac{\binom{5}{4}4!\binom{5}{1}5!}{10!} = \frac{5}{252} \\
P\{X=6\} &= \frac{\binom{5}{5}5!\binom{5}{1}4!}{10!} = \frac{1}{252} \\
P\{X=7\} &= P\{X=8\} = P\{X=9\} = P\{X=10\} = 0
\end{aligned}
$$

Problem 5 The possible values are $n, n-2, n-4, ..., -n+4, -n+2, -n$.

Problem 13 Let X be the total dollar value of all sales. Then X can take the values $0, 500, 1000, 1500, 2000$, and we have

$$
\begin{aligned}
P\{X=0\} &= 0.7 \cdot 0.4 = 0.28 \\
P\{X=500\} &= \frac{1}{2}(0.3 \cdot 0.4 + 0.7 \cdot 0.6) = 0.27 \\
P\{X=1000\} &= \frac{1}{2}(0.3 \cdot 0.4 + 0.7 \cdot 0.6) + \frac{1}{4}0.3 \cdot 0.6 = 0.315 \\
P\{X=1500\} &= 2\frac{1}{4}0.3 \cdot 0.6 = 0.09 \\
P\{X=2000\} &= \frac{1}{4}0.3 \cdot 0.6 = 0.045
\end{aligned}
$$

Problem 14

$$
\begin{aligned}
P\{X=0\} &= \frac{0!}{2!} = \frac{1}{2} \\
P\{X=1\} &= \frac{1!}{3!} = \frac{1}{6} \\
P\{X=2\} &= \frac{2!}{4!} = \frac{1}{12} \\
P\{X=3\} &= \frac{3!}{5!} = \frac{1}{20} \\
P\{X=4\} &= \frac{4!}{5!} = \frac{1}{5}
\end{aligned}
$$

Problem 17 (a)

$$
\begin{aligned}
P\{X=1\} &= P\{X \le 1\} - P\{X < 1\} = \frac{1}{2} - \frac{1}{4} = \frac{1}{4} \\
P\{X=2\} &= \frac{11}{12} - \frac{3}{4} = \frac{1}{6} \\
P\{X=3\} &= 1 - \frac{11}{12} = \frac{1}{12}
\end{aligned}
$$

(b) $P\left\{\frac{1}{2} < X < \frac{3}{2}\right\} = \frac{5}{8} - \frac{1}{8} = \frac{1}{2}$.

Problem 19

$$
\begin{aligned}
P\{X=0\} &= \frac{1}{2} \\
P\{X=1\} &= \frac{3}{5} - \frac{1}{2} = \frac{1}{10} \\
P\{X=2\} &= \frac{4}{5} - \frac{3}{5} = \frac{1}{5} \\
P\{X=3\} &= \frac{9}{10} - \frac{4}{5} = \frac{1}{10} \\
P\{X=3.5\} &= 1 - \frac{9}{10} = \frac{1}{10}
\end{aligned}
$$