1. Suppose that \( (X_n)_{n=0}^{\infty} \) is Markov(\( \lambda, P \)). Define \( Y_n = X_{kn} \) for some \( k \geq 1 \). Show that \( (Y_n)_{n=0}^{\infty} \) is Markov(\( \lambda, P^k \)).

2. We will consider a series of Markov chains which are given by a unidirectional ring with one escape point, i.e. pick \( N > 1 \) and \( 0 < p < 1 \), and consider the Markov chain with \( I = \{1, 2, \ldots, N + 1\} \) and transition probabilities

\[
\begin{align*}
    p_{1,1} &= 0, \quad p_{1,2} = p, \quad p_{1,N+1} = 1 - p, \\
    p_{i,i+1} &= p, \quad p_{i,i} = 1 - p, \quad i = 2, \ldots, N - 1, \\
    p_{N,1} &= p, \quad p_{N,N} = 1 - p, \\
    p_{N+1,N+1} &= 1.
\end{align*}
\]

Some examples are:

(We are not drawing in the loops in these diagrams, since the weights are implied.)

Denote \( A = \{N + 1\} \).

Prove:

(a) \( N + 1 \) is an absorbing state;
(b) If \( 0 < p < 1 \), then \( k_i^A(p) = 1 \) for all \( i = 1, \ldots, N + 1 \).
(c) If \( p = 0 \), then \( k_i^A(p) = 0 \) for all \( i = 2, \ldots, N \), and \( k_1^A(p) = h_{N+1}^A(p) = 1 \).
(d) Compute \( k_i^A(p) \) for all \( i \) and all \( 0 \leq p \leq 1 \). For which \( i \) is this lowest? Highest? Does this make sense?
(e) Show that \( k_1^A(\cdot) \) is discontinuous at 0, i.e. that

\[
    k_1^A(0) \neq \lim_{p \to 0^+} k_1^A(p).
\]

Explain this paradox.
3. Show that \( P_i, E_i \) satisfy the same formulas as \( P, E \) do in the LTP, LTE. More specifically, if \( \{B_k\} \) is a partition of \( \Omega \), show that

\[
P_i(A) = \sum_k P_i(A|B_k)P_i(B_k), \quad E_i[A] = \sum_k E_i[A|B_k]P_i(B_k).
\]

4. Show that, for any \( i, j \in I, A \subseteq I, \) and \( i \notin A \),

\[
P_i(H^A < \infty | X_1 = j) = P_j(H^A < \infty),
\]

and

\[
E_i[H^A|X_1 = j] = 1 + E_j[H^A].
\]

5. Here we systematically develop the solution of the recursion formula for \( h_i \):

\[
h_i = ph_{i+1} + qh_{i-1}, \quad h_0 = 1.
\]  (1)

(a) Show that any constant solution \( h_i = A \) satisfies (1).

(b) Show that solution \( h_i = B \left( \frac{q}{p} \right)^i \) satisfies (1).

(c) Show by linearity that

\[
h_i = A + B \left( \frac{q}{p} \right)^i
\]  (2)
does as well.

(d) Show that the general solution to (1) has at most two free parameters, so all solutions are of the form in (2).

6. Consider the recurrence relation

\[
x_{n+1} = Ax_n + B, \quad x_0 = C.
\]  (3)

The goal of this problem is to prove that

\[
x_n = \left( C - \frac{B}{1-A} \right) A^n + \frac{B}{1-A}.
\]

To see this, first consider two solutions to the relation (3) with different initial conditions, for example

\[
x_{n+1} = Ax_n + B, \quad x_0 = C, \quad y_{n+1} = Ay_n + B, \quad y_0 = C'.
\]

What recurrence relation does \( z_n = x_n - y_n \) satisfy? Solve this relation explicitly. Now, is there a choice of \( C' \) that makes the solution \( y_n = \) constant?
7. Consider the transition matrix

\[ P = \frac{1}{2} \begin{pmatrix} 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{pmatrix}. \]

Draw the graph corresponding to this Markov chain, and compute \( p_{ij}^{(n)} \) for all \( i, j \in [4] \) and \( n \geq 0 \). Describe in words what happens to probability as a dynamical process in time.

8. Consider the transition matrix

\[ P = \frac{1}{3} \begin{pmatrix} 0 & 1 & 1 & 1 \\ 1 & 1 & 1 & 0 \\ 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{pmatrix}. \]

Draw the graph corresponding to this Markov chain, and compute \( p_{ij}^{(n)} \) for all \( i, j \in [4] \) and \( n \geq 0 \). Describe in words what happens to probability as a dynamical process in time.